

Inflation securities valuation with macroeconomic-based no-arbitrage dynamics

Gabriele Sarais
Department of Mathematics
Imperial College London
gs1608@imperial.ac.uk

Damiano Brigo
Department of Mathematics
Imperial College London
damiano.brigo@imperial.ac.uk
Capco Institute

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Abstract

We develop a model to price inflation and interest rates derivatives using continuous-time dynamics that have some links with macroeconomic monetary DSGE models equipped with a Taylor rule: in particular, the reaction function of the central bank, the bond market liquidity, inflation and growth expectations play an important role. The model can explain the effects of non-standard monetary policies (like quantitative easing or its tapering) and shed light on how central bank policy can affect the value of inflation and interest rates derivatives.

The model is built under standard no-arbitrage assumptions. Interestingly, the model yields short rate dynamics that are consistent with a time-varying Hull-White model, therefore making the calibration to the nominal interest curve and options straightforward. Further, we obtain closed forms for both zero-coupon and year-on-year inflation swap and options. The calibration strategy we propose is fully separable, which means that the calibration can be carried out in subsequent simple steps that do not require heavy computation. A market calibration example is provided.

The advantages of such structural inflation modelling become apparent when one starts doing risk analysis on an inflation derivatives book: because the model explicitly takes into account economic variables, a trader can easily assess the impact of a change in central bank policy on a complex book of fixed income instruments, which is normally not straightforward if one is using standard inflation pricing models.

Keywords: Inflation, Derivatives, DSGE Models, Monetary Macroeconomic Models, Calibration, Hull-White Model, Central Bank Policy, Risk-Neutral Valuation, Option Pricing, Taylor Rule, Inflation-Linked Securities, Stress Testing, Macro-Hedging.

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1 Monetary macroeconomic inflation models

We consider a macroeconomic model and propose a strategy to use it to price inflation derivatives: the main advantage of this approach is that the inflation dynamics are not taken exogenously but rather are the result of a well-established macroeconomic model with central bank policy. In particular, the co-movement of inflation and nominal interest rates is not taken as an input or modelled via a correlation structure (as it happens in many models currently used in the industry) but is the result of central bank policy, via a well-known macroeconomic relationship (Taylor rule or some variation of it).

The task is not straightforward because most macroeconomic literature is written in a somewhat less formalised way compared to financial mathematics. Expectations are taken only with respect to the real-world econometric measure (which is known as \mathbb{P} or as “physical measure” in financial mathematics): therefore there is no apparent need to specify the measure used to take expectations, and measure changes are not used. No mention is made of filtrations, adapted processes, measurability. Distributional assumptions tend to be loose (randomness is usually introduced via some so-called “white noise”, defined as a zero-mean process whose realisations are independent from each other over time). Stochastic processes tend to be assumed to reach a “steady state”, i.e. to converge to some equilibrium value in the long run: this level is always supposed to exist and to be finite. Sometimes variables are expressed as their percentage deviations with respect to their long term equilibrium level. Securities payoffs may be defined with few details.

With all these issues, this theory is the one that central bankers, economists, researchers, and market operators use and refer to: it can not be ignored. The challenges that one faces to use this theory in financial mathematics to carry out derivatives pricing are manifold: to complement the macroeconomic model with all the mathematical machinery that has been originally taken as a given in a way that the kernel of the model is not arbitrarily changed but is rather enhanced by an improved formalism. Further, when changes are made to the original model assumptions, these changes have to be not invasive and have to bear a clear

advantage, especially in the calibration phase. At the same time, some approximations may be needed to derive some results that are essential for pricing (closed forms for nominal and inflation bonds, for example).

The article is structured as follows. Firstly, we build a general axiomatic framework around the original macroeconomic discrete-time model: this entails specifying the feature of the time scale, probability space, and traded instruments. Then, for the benefit of the reader not expert in monetary macroeconomics, all the economic quantities and assumptions are listed and defined. Secondly, we introduce a standard monetary macroeconomics discrete-time model (DSGE model, or “Dynamic Stochastic General Equilibrium” model) where some \mathbb{P} -dynamics for inflation are derived from optimality conditions and realistic market frictions. Thirdly, we derive the expression for the nominal rate and inflation rate volatility and higher order moments based on the DSGE model: they turn out to be linear combinations for the volatilities and the higher order moments of the random processes used originally in the DSGE model; the advantage is clear, as one can choose these parameters to match the moments of the distribution implied by market-traded options on interest rates and inflation. Fourthly, we obtain approximated expressions for the nominal and inflation bonds, that allow one to calibrate the model to the observed nominal and inflation term structures.

To sum up, the first part of this article constitutes a useful attempt to bridge the gap between monetary macroeconomics and financial mathematics: as a result, we have built a toy pricing model based on the DSGE model. However, in the second part of this article we suggest how the framework can be somewhat translated into continuous time to improve its analytical tractability and to take into account some very recent market features, like low interest rates and quantitative easing. Although there is no exact correspondence between the original discrete-time DSGE model and the newly-introduced continuous-time dynamics, the latter are clearly inspired by the former. We develop the theory in continuous time under no-arbitrage, and provide some closed forms for common inflation and interest rates payoffs. Finally, we make an example of calibration of the continuous-time model to market data. To conclude, some pricing and risk management applications are discussed.

2 Introduction to DSGE models

DSGE models are an essential tool for the working macro-economist: they are widely used both in academia and by central banks since they explain the short term real effects of monetary policy. There is strong empirical evidence supporting the idea that money has real effects: DSGE models describe this effect by assuming a stochastic environment, optimizing behaviour and nominal rigidities in the economy. Consumers maximize their expected utility, which is based on consumption and real cash balances; firms maximize their expected profit stream but are not able to change in each period the prices they charge. The result is a discrete-time model where the macroeconomic variables are affected by their future expectations and some external shocks. The short term nominal interest rate (“short rate”) is part of these dynamics. A further assumption is that the central bank uses a Taylor rule to set the short rate: this means that the short rate is changed in response to the other macroeconomic variables using a simple linear rule (see Taylor [40]). This approach, albeit simple, has proven to be powerful to explain the central bank behaviour.

Finally, we stress that so far we referred to DSGE models in plural as they can be regarded as a family of models that share the main features listed above: consumer habits, capital, labour market rigidities, government, taxes, lagged variables, different central bank policies can be introduced in this framework, giving rise to more complex dynamics. Here we describe the baseline version of this model, which offers enough flexibility for our purposes.

We present the assumptions of a basic version of the DSGE macroeconomic model, which explains the behaviour of the inflation rate p_i and the output gap x_i based on a general description of the economy. A complete description of this model can be found in this section or in Walsh [42], which we follow to present the model. However, before presenting the macroeconomic model, we specify the axiomatic foundations that are implicit in the model and that are normally not fully specified by economists.

2.1 Axiomatic foundations

2.1.1 Time scale

The model is set in discrete time, where time is a non-negative variable: $t_i \in \mathbb{T} = \{t_0, t_1, t_2, \dots, t_n, \dots\}, n \in \mathbb{N}$.

Here t_0 is the present time. To preserve generality, the discrete-time points are not required to be equally spaced. For a variable y at time t_i we often write y_i to make the notation lighter: similarly, the discrete-time stochastic processes $\{y_{t_i}\}_{i=0,1,\dots}$ is denoted by $\{y_i\}$.

2.1.2 Probability space

We work with the probability triplet $\{\Omega, \mathcal{F}, \mathbb{P}\}$ and assume the existence of a market filtration $\{\mathcal{F}_{t_i}\}_{i \geq 0}$. In particular \mathbb{P} is the real-world (“physical”) probability measure. All filtration-related concepts (mainly the martingality property) are defined with respect to the market filtration. To simplify the notation, in discrete time we use the following notation for conditional expectations: $\mathbb{E}_i x_{i+j} = \mathbb{E}_i[x_{i+j}] = \mathbb{E}^{\mathbb{P}}[x_{t_{i+j}} | \mathcal{F}_{t_i}]$.

Therefore, in this section if no probability measure is specified, the expectation is taken with respect to the real-world measure (\mathbb{P}). To perform a measure change from the physical measure \mathbb{P} to the risk-neutral measure \mathbb{Q} , we introduce the Radon-Nikodym derivative $(d\mathbb{Q}/d\mathbb{P})|_{t_i}$, written μ_i for brevity. All regularity requests for the measure change process to exist and to be an L^2 positive martingale are supposed to hold.

2.1.3 Financial instruments

We assume that the financial market is such that there are no transaction costs nor taxes: investors can take any position (either long or short) in any asset. We assume the existence of the following:

1. The short term nominal interest rate n_i – set by the central bank – is the interest agreed at time t_{i-1} and paid at time t_i by the bank account on the balance at time t_{i-1} ¹. It is used to discount payments. The short term nominal interest rate process $\{n_i\}_{i=0,1,\dots}$ is a previsible process, i.e. the short term nominal interest rate n_i is \mathcal{F}_{i-1} -measurable.
2. The bank account $B_i = \prod_{j=1}^i (1 + \tau_j n_j)$, with $B_0 = 1$. Here τ_i represents the year fraction between times t_{i-1} and t_i . Since the interest rate n_i is \mathcal{F}_{i-1} -measurable, the bank account process $\{B_i\}_{i=0,1,\dots}$ is a previsible process. At time t_{i-1} the cash flow that will occur at time t_i is already known: this is why the bank account is often referred to as the riskless asset.
3. The first two properties implicitly imply a lower bound on negative short nominal interest rates: in this model rates can be negative (as they have been in 2012, for example German Bunds up to 2 years maturity), however they can't be lower than -100%, otherwise the nominal bank account would have a negative value, which is not possible. Rational agents would not put money into such account that turns assets into liabilities.
4. A system of discount bonds $P(t_i, t_N)$, that pay one unit of currency at time t_N and have the following properties:

- $P(t_i, t_N) = \mathbb{E}_i^{\mathbb{Q}} \left[\prod_{j=i+1}^N (1 + \tau_j n_j)^{-1} \right]$
- $P(t_i, t_i) = 1, \forall i$
- $P(t_i, t_N) > 0, \forall i \leq N$
- $\lim_{N \rightarrow +\infty} P(t_i, t_N) = 0$
- $P(t_i, t_{i+1}) = B_i / B_{i+1}$.

¹Here we are assuming that the central bank lends money to the commercial banks at the same interest rate paid by these to the money market account holders. We are making the simplifying assumption that the central bank reviews its interest rate with the same time scale by which the interest are accrued in the money market account. This assumption allows one to include in the pricing model a fairly realistic description of central policy.

5. The seasonality-adjusted price index process $\{I_i\}_{i=0,1,\dots}$ that describes the evolution of the price level².
6. A system of zero-coupon inflation index swaps (ZCIIS), such that the floating leg pays $(I_{i+M-1}/I_i) - 1$ and the fixed leg pays $(1 + X_M)^{M\tau_i} - 1$. The strikes X_i are quoted at time t_i for all maturities $t_M > t_i$. Inflation payments are time-lagged in this model as it happens in reality: in fact the price index is subject to revisions and in practice ZCIIS pay the inflation lagged by one period.
7. A system of index-linked zero-coupon bonds $P^I(t_i, t_M)$, which pay at maturity t_M the cash equivalent of the price index I_{M-1} . These bonds are priced consistently with the zero coupon inflation swaps seen in the previous point. Inflation payments are time-lagged in this model as it happens in reality: in fact the price index is subject to revisions and in practice the inflation bonds pay the inflation lagged by one period. These bonds are quoted at time t_i for all maturities $t_M > t_i$.

Macroeconomic variables. The inflation rate is defined by $p_i = ((I_i/I_{i-1}) - 1)/\tau_i$. This is the annualised percentage growth rate of the price index.

The output gap x_i is defined as the difference between the actual and the potential log-linearised growth rate of the economy³: $x_i = \hat{y}_i - \hat{y}_i^f$. To provide a complete definition of the output gap, we introduce the Gross Domestic Product (GDP) Y_i – also known as output – which is the value of all final goods and services produced in the economy between times t_{i-1} and t_i . The GDP annualised growth rate is defined as: $y_i = ((Y_i/Y_{i-1}) - 1)/\tau_i$. The growth rate y_i is assumed to have a long term equilibrium level \bar{y} such that $\mathbb{E}(y_i) \rightarrow \bar{y}$ as $i \rightarrow +\infty$. The variable \hat{y}_i is defined as the percentage deviation between the GDP growth rate y_i and its long term equilibrium level \bar{y} : $\hat{y}_i = ((y_i/\bar{y}) - 1)$. Economists often refer to it as the log-linearised GDP growth rate, as $\hat{y}_i = ((y_i/\bar{y}) - 1) \cong \log(y_i/\bar{y})$.

If we assume that the economy is subject to some “inefficiencies”, we can introduce the potential GDP Y_i^f , which can be defined as the GDP produced if there is no inefficiency: intuitively these inefficiencies prevent the actual GDP Y_i from reaching the “full employment” GDP Y_i^f . Therefore we similarly derive the variables y_i^f , \bar{y}_i^f , and \hat{y}_i^f , which complete the definition of the output gap x_i .

We assume that the processes $\{Y_i\}_{i=0,1,\dots}$, $\{Y_i^f\}_{i=0,1,\dots}$, and $\{I_i\}_{i=0,1,\dots}$ are adapted, therefore also the processes $\{x_i\}_{i=0,1,\dots}$ and $\{p_i\}_{i=0,1,\dots}$ are adapted. To complete the formalisation, one needs to assume that all stochastic processes involved in the model converge in some sense at a finite equilibrium level when time tends to infinity. No further specification of such convergence is normally made in the macroeconomic model.

Economic assumptions. We list the microeconomic and macroeconomic assumptions:

1. The economy is closed, i.e. there is no exchange rate nor foreign market.
2. All markets are in equilibrium, i.e. demand matches supply for all goods and services markets.
3. The economy is a monetary one, i.e. there is no barter.
4. The representative consumer maximizes his utility function under an intertemporal budget constraint.
5. The representative consumer draws his utility from consuming and keeping cash balances for safety (money in utility approach).
6. There is no public sector, therefore there is no taxation.
7. Labour is the only production factor: no capital is required, therefore there are no investments.
8. Savings are invested in bonds that pay a nominal interest rate.

²Price indices time series clearly show seasonality, mainly driven by sales in January and July and prices increases around Christmas. we do not model these patterns directly at this stage because a seasonality correction can be easily introduced at the last stage. This can be done by assuming that the monthly inflation rate differs from the seasonality-adjusted inflation rate by a certain percentage. Intuitively, seasonality is more relevant for short-maturity inflation trades.

³The reason why we are involving log-linearisation will become clear shortly. More information is also available later in this section.

9. The representative consumer consumes multiple goods, each produced in a monopolistic market.
10. The output coincides with the private consumption, as there is no government expenditure, no import/export, no taxes nor investment.
11. Firms maximize profits but are not free to modify in each period the prices they charge (sticky prices).
12. The central bank sets the short rate as a linear function of inflation and output gap (Taylor rule). The short rate moves around its equilibrium level.
13. The short rate can be negative in some circumstances.
14. There exists a system of expectations for the output gap and inflation.
15. Time is discrete.
16. There is no credit risk.

2.2 Model derivation

We follow Walsh [42] to introduce the main equations of the DSGE baseline model: as the material of this section is standard, we give a high level overview. Another interesting overview can be found in Clarida, Gali & Gertler [13]. The reader that is not interested can skip this section.

2.2.1 Economy description

The baseline model we work with represents a simple closed economy, with no government and no tax system. The production function depends only on labour since capital is not considered: therefore there is no investment. From a macroeconomic perspective we can therefore state that the output at time t_i equals the aggregate consumption at time t_i :

$$Y_i = C_i.$$

The economy is a monetary one with money M_i and price level I_i .

2.2.2 Consumers

The representative household solves a two-steps optimisation problem. It first decides how to allocate its total consumption between different goods – produced by monopolistically competitive final goods producers (firms) – and then chooses how much to consume in total, how much cash to hold, how much to invest in bond holdings and how many hours to work.

In the first step we assume the existence of a continuum of goods c_j produced by a continuum of monopolistic firms j ($j \in [0, 1]$). At time t_i the household chooses the combination of goods c_{ji} that minimizes the cost of the total consumption:

$$\min \int_0^1 p_{ji} c_{ji} dj$$

by taking into account the constraint

$$\left(\int_0^1 (c_{ji})^{\frac{\theta-1}{\theta}} dj \right)^{\frac{\theta}{\theta-1}} \geq C_i.$$

Here p_{ji} is the price of the good j at time t_i and C_i is the total consumption time t_i . The parameter θ is used to model the price elasticity, i.e. how price-sensitive consumption is. This standard optimisation problem

is solved in Walsh [42] (p. 233) and yields the optimal amount of consumption of good j given the general price level I_i , the total consumption C_i (to be determined in the next step) and the price of good j , p_{ji}

$$c_{ji} = \left(\frac{p_{ji}}{I_i} \right)^{-\theta} C_i. \quad (1)$$

The second step is modelled as an intertemporal maximisation of the expected utility under a budget constraint, and yields the usual Euler conditions.

The representative household draws its utility from consuming (C_i) and holding real cash balances (M_i/I_i) as insurance against uncertainty: furthermore it has negative utility from supplying labour N_i and can save money and purchase bonds B_i that pay a nominal interest rate n_i in each period. We assume a power utility function: the problem is therefore to find the sequences C_i , M_i , B_i and N_i that solve the problem

$$\max \sum_{t_i=t_0}^{\infty} \beta^{t_i} \mathbb{E}_0 \left[\frac{C_i^{1-\sigma}}{1-\sigma} + \frac{\alpha}{1-d} \left(\frac{M_i}{I_i} \right)^{1-d} - \frac{N_i^{1+\eta}}{1+\eta} \right].$$

The parameters σ , d , $\alpha > 0$, η indicate how consumption, real cash balance and labour supply influence the utility function. The expectation $\mathbb{E}[\cdot]$ is taken with respect to the physical measure \mathbb{P} , as usual in any macroeconomic model: in this section when no measure is specified it is assumed that the physical measure is used. The parameter $\beta \in (0, 1]$ represents as usual a subjective discount factor over one period. The parameter σ , that is also known as “relative risk aversion”, is used to model elasticity of utility to consumption in a constant relative risk aversion (CRRA) utility function. When σ is very high, the agents are extremely risk-averse, as an increase in consumption creates a smaller increase in utility than the correspondent reduction in utility given the same absolute reduction in consumption; when σ is zero, there is risk-neutrality, i.e. the utility grows linearly with consumption; when σ tends to 1, the utility function becomes a logarithmic utility function, which is moderately risk averse. The optimisation is carried out under the constraint that the total wealth at time t_i (which is allocated between consumption, real cash balance and bond holding) has been derived from the previous period or gained from supplying labour (W_i is the wage gained for 1 unit of labour at time t_i). No wealth is introduced into the system *ex nihilo*:

$$C_i + \frac{M_i}{I_i} + \frac{B_i}{I_i} = \frac{W_i N_i}{I_i} + \frac{M_{i-1}}{I_i} + \frac{B_{i-1}}{I_i} (1 + n_{i-1}).$$

The derivation of the Euler conditions is standard and can be found for example in the second chapter of Walsh [42]. The first order conditions for this problem are the following:

$$C_i^{-\sigma} = (1 + n_i) \beta \mathbb{E}_i \left[\frac{I_i}{I_{i+1}} \right] C_{i+1}^{-\sigma} \quad (2)$$

$$\alpha \left(\frac{M_i}{I_i} \right) C_i^{\sigma} = \frac{n_i}{1 + n_i}. \quad (3)$$

$$\frac{N_i^{\eta}}{C_i^{-\sigma}} = \frac{W_i}{I_i}. \quad (4)$$

Since we assume that there is no government, no capital stock (and therefore no investment) and the economy is closed, we substitute the consumption with the output definition $Y_i = C_i$, getting

$$Y_i^{-\sigma} = (1 + n_i) \beta \mathbb{E}_i \left[\frac{I_i}{I_{i+1}} \right] Y_{i+1}^{-\sigma}.$$

This condition may be rewritten in log-linearized terms around a zero inflation equilibrium point, after

making some approximations:

$$\hat{y}_i = \mathbb{E}_i \hat{y}_{i+1} - \frac{1}{\sigma} (\hat{n}_i - \mathbb{E}_i p_{i+1}). \quad (5)$$

The inflation rate p_i is defined as the annualised relative change of the price level I_i from t_{i-1} to t_i .⁴

2.2.3 Firms

The firm profit maximisation problem has to take into account three constraints: the demand curve, the production technology and price stickiness. It involves finding the optimal amount of labour N_i to minimize the production cost and the optimal good price p_{ji} to maximize the expected profit stream. The demand curve is (1). Secondly, technology is such that the output of the j -th firm depends only on labour N_{ji}

$$c_{ji} = Z_i N_{ji}.$$

Z_i is a positive random variable with mean 1 that represents a stochastic productivity shock. Thirdly, firms are able to adjust their prices in each period only with probability $1 - \omega$. This price stickiness assumption is the most interesting one and is essential to define the inflation dynamics of this model.

The first consequence is that the output Y_i will deviate from the output in flexible prices Y_i^f : by making use of (4), we can then define their difference in log-linearized terms as the output gap

$$x_i = \hat{y}_i - \hat{y}_i^f.$$

We will not explain the subsequent details: instead we develop some intuition of the inflation mechanics. Since prices are sticky and firms are maximizing their expected profit stream, firms will tend to increase their prices not only if production costs rise (which would also happen in a flexible prices framework), but also to compensate for the expected losses they can face as they may not increase prices in the future (with probability ω).

This has two important consequences: firstly, as prices influence output via the demand curve (1) and the macroeconomic identity $Y_i = C_i$, inflation will be related to the output gap. The output gap increases with inflation. Secondly, if there are inflation expectations, firms will raise prices in the current period because they may not be able to do so in the future. Inflation is therefore a self-fulfilling prophecy.

The result, after some algebraic manipulations, is the so-called neo-Keynesian Phillips curve, which states that the current level of inflation depends both on inflation expectations and the output gap:

$$p_i = \beta \mathbb{E}_i p_{i+1} + k x_i. \quad (6)$$

The parameter $k \geq 0$ can be regarded as a measure of the market price flexibility and is defined as

$$k = \frac{(1 - \omega)(1 - \beta\omega)(\sigma + \eta)}{\omega}.$$

It is worth stressing that if prices never change, $\omega = 1$: therefore k equals zero and inflation will only be driven by expectations. As before, the derivation can be found in Walsh [42] (5.4, 5.7).

⁴It is worth explaining how the log-linearisation used above works. Given a variable F_i at time t_i we assume that its long term equilibrium level is \bar{F} (i.e. that the limit of F_i when time goes to infinity is \bar{F}). With the lower case hat \hat{f}_i we indicate the deviation at time t_i of the variable F_i from its long term equilibrium level \bar{F} in percentage terms: this can be approximated with the natural logarithm of their ratio for small deviations. In formulas:

$$\hat{f}_i = \frac{F_i}{\bar{F}} - 1 \cong \log\left(\frac{F_i}{\bar{F}}\right).$$

Uhlig [41] gives extensive explanations and examples of this technique: given the analogy between this transformation and the natural logarithm, products can be approximated with sums, powers become multiplicative coefficients and constants disappear.

2.2.4 Putting things together

Equation (5) can be rewritten in terms of output gap $x_i = \hat{y}_i - \hat{y}_i^f$: defining

$$u_i = \mathbb{E}_i \hat{y}_{i+1}^f - \hat{y}_i^f,$$

we get to a final form for (5) that can be put in a system with (6)

$$x_i = \mathbb{E}_i x_{i+1} - \frac{1}{\sigma} (\hat{n}_{i+1} - \mathbb{E}_i p_{i+1}) + u_i. \quad (7)$$

As we define the rate n_{i+1} as the rate set by the central bank at time t_i and paid at time t_{i+1} we have written \hat{n}_{i+1} rather than \hat{n}_i . This curve can be interpreted as a neo-Keynesian demand curve, where the output gap shows a negative dependency on a function of the real interest rate. The process $\{u_i\}_{i=0,1,\dots}$ can be thought as a discrete-time stochastic process that relates the level of the log linearised flexible price output deviation from its expectations: this difference should depend somehow on the productivity shock Z_i , but for our purposes we can think of it as a general stochastic process. Again, we stress that the original macroeconomic model does not make any further assumption on the shock processes: we will make the necessary assumptions this problem in the following sections, where the DSGE model is used for pricing purposes.

2.2.5 The Taylor rule and the central bank

Equations (7) and (6) define a discrete-time, bi-dimensional, forward looking stochastic system which is influenced by two exogenous variables: the log-linearized short rate \hat{n}_{i+1} and the process $\{u_i\}_{i=0,1,\dots}$, related to the productivity shock. We introduce the central bank, which uses the short rate as policy instrument. In each period the central bank changes the short rate in response to the inflation and output gap with the following rule:

$$\hat{n}_{i+1} = \delta_\pi p_i + \delta_x x_i + v_i. \quad (8)$$

This rule, proposed by Taylor [40], states that the central bank responds to inflation and output gap by setting the short rate: a discrete-time stochastic process $\{v_i\}_{i=0,1,\dots}$, independent from the process $\{u_i\}_{i=0,1,\dots}$, is added to increase the flexibility of the model. We remind the reader that the rate n_{i+1} is set by the central bank at time t_i and paid at time t_{i+1} : for this reason we allow a lag in the above form of the Taylor rule. At this stage we also notice that the short rate can be negative in this formulation, which is consistent with the assumptions we have made earlier: however values of the nominal rate below -100%, albeit theoretically possible, are to be ruled out under a reasonable model parametrisation. Finally one notes that the Taylor rule has been defined for \hat{n}_{i+1} , which, as explained for the other variables, is the percentage deviation of the nominal rate from its equilibrium level.

Bullard & Mitra [11] analyse similar rules with more realistic timing assumptions (the central bank may be reacting to future expectations of gap and inflation, or may be looking at their lagged values instead). In addition, the short rate can be smoothed as suggested by Woodford [43], essentially by combining (8) with an autoregressive process. This framework is somehow simple, as the central bank is not optimizing any objective function: however, it has explained the behaviour of the FED in the last decades, as shown by Clarida, Gali & Gertler [14]. Finally, this linear rule can be regarded as good linear approximation of the optimal policy solution.

2.3 System stability

If the Taylor rule (8) is plugged into (7) and (6), we obtain the following system:

$$\begin{bmatrix} x_i \\ p_i \end{bmatrix} = \frac{1}{\sigma + \delta_x + k\delta_\pi} \left(\begin{bmatrix} \sigma & 1 - \beta\delta_\pi \\ k\sigma & k + \beta(\sigma + \delta_x) \end{bmatrix} \mathbb{E}_i \begin{bmatrix} x_{i+1} \\ p_{i+1} \end{bmatrix} + \begin{bmatrix} 1 \\ k \end{bmatrix} (\sigma u_i - v_i) \right) \quad (9)$$

The notation is made more compact by defining:

$$A = \frac{1}{\sigma + \delta_x + k\delta_\pi} \begin{bmatrix} \sigma & 1 - \beta\delta_\pi \\ k\sigma & k + \beta(\sigma + \delta_x) \end{bmatrix}$$

$$K = \frac{1}{\sigma + \delta_x + k\delta_\pi} \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$\xi_i = \begin{bmatrix} x_i \\ p_i \end{bmatrix}$$

$$w_i = (\sigma u_i - v_i)$$

Using the above definitions we get a more compact expression of the system:

$$\xi_i = A\mathbb{E}_i\xi_{i+1} + Kw_i. \quad (10)$$

We investigate the stability conditions, which is equivalent to ask what reaction function – characterised by the parameters δ_π and δ_x – keeps the economy on a stable path. For example, if the central bank only responds to inflation (i.e. $\delta_x = 0$), we ask whether δ_π has to be greater or lower than one, i.e. if the central bank has to increase the short rate above its equilibrium level more or less than the realised inflation. Clarida, Gali & Gertler [14] show that $\delta_\pi > 1$ is typical of the FED during the Volker tenure (in the early 1980s in the U.S.), which was characterized by lower inflation and output volatility: this is confirmed by simulations.

The economic intuition is that a reaction parameter close to one means that the nominal rate is increased by the same amount of inflation, thus keeping the real rate unchanged and not stimulating the economy. Bullard & Mitra [11] find that in general the system is stable if and only if

$$k(\delta_\pi - 1) + (1 - \beta)\delta_x > 0. \quad (11)$$

They obtain this rule by requiring that both eigenvalues of A lie inside the unit circle. Alternatively, by setting $B = A^{-1}$ the deterministic part of system (10) can be rewritten as

$$\mathbb{E}_i\xi_{i+1} = B\xi_i.$$

In this case the request is that both eigenvalues of B lie outside the unit circle: this equivalent request is derived by Blanchard & Khan [6] and used by Flashel & Franke [21] or Walsh [42].

3 Using the DSGE for pricing purposes

3.1 Arbitrage-free pricing

The set-up introduced is standard (we have followed Walsh [42]), however it allows one to use a DSGE macroeconomic model to price inflation derivatives in a no-arbitrage framework with a few minor changes. In general, the pricing kernel ψ_i properties discussed for example in Constantinides [16] enable one to write the present value at time t_i of a derivative h_i paying the inflation-linked payoff H_N^π at time t_N in the form:

$$h_i = \mathbb{E}^\mathbb{P}[\psi_N H_N^\pi | \mathcal{F}_i] \frac{1}{\psi_i}.$$

Here we build a toy pricing model in discrete time that is based on the DSGE model.

3.1.1 Use of the macroeconomic model: inputs and outputs

We make a distinction between input parameters (the structural parameters of the DSGE model, equilibrium nominal rates, inflation expectations, output-gap expectations), and calibrated parameters (the volatilities and the market prices of risk).

Calibrating the market prices of risk is not a usual procedure in derivatives pricing, because the real-world drift is not an input in the classic Black-Scholes framework to price contingent claims: however, the DSGE model takes expectations (under the \mathbb{P} measure) as an input. Since these expectations play the role of the drift in (10) we need to take both inflation expectations and market implied levels (from the zero-coupon inflation swaps, for example) to calibrate the market prices of risk. The expectation of inflation is a kind of self-fulfilling prophecy: if there are expectations of inflation, then inflation will tend to rise. This exercise is particularly useful for inflation markets, since it is often observed that inflation forecasts and expectations can significantly differ from levels of inflation calculated on a forward basis. Such differences arise both because of risk aversion and market supply and demand factors: the market is to a significant extent a “one-way street”, overall “short” inflation. In other words market participants on the whole wish to hedge themselves against rises of inflation. In particular, pension funds liabilities have to be covered.

The idea of using market forecasts as model input, although not commonly used in standard derivatives pricing, lets one to use a theoretically consistent macroeconomic model for the pricing of inflation derivatives.

The algorithm we suggest calibrates to both the nominal term structure and the zero-coupon inflation index swaps (ZCIIS), leaving much flexibility to calibrate to market smiles. To achieve this we explore the statistical properties of the main economic variables, as implied by the DSGE model presented above.

3.1.2 Statistical properties of the inflation rate

From equation (10) we write explicitly the dynamics of the inflation rate:

$$p_i = A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} + K_2 w_i. \quad (12)$$

Here $A_{i,j}$ is the (i,j) -th element of the matrix A . This equation states that the inflation dynamics depend on future expectations of output gap and inflation, plus a stochastic noise term introduced by the dynamics of the output gap and the central bank behaviour: we can safely assume that other factors, such as measurement errors, price index basket reshuffle or any other idiosyncratic factor not directly modelled in this framework, may add noise to the inflation dynamics.⁵ On the basis of these considerations, we add a further independent source of randomness, modelled with the adapted process $\{z_i\}_{i=0,1,\dots}$: we require this process to have zero mean, to be independent from its past realisations, to be independent from $\{u_i\}_{i=0,1,\dots}$ and $\{v_i\}_{i=0,1,\dots}$, and to have finite variance $\text{Var}(z_i)$, third and fourth moments ($\text{Skew}(z_i)$ and $\text{Kurt}(z_i)$ respectively).

The new expression for the inflation rate becomes:

$$p_i = A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} + K_2 w_i + z_i. \quad (13)$$

Its mean, variance and autocovariance are therefore:

$$\begin{aligned} \mathbb{E}[p_i] &= A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} \\ \text{Var}(p_i) &= (K_2)^2(\sigma^2 \text{Var}(u_i) + \text{Var}(v_i)) + \text{Var}(z_i) \\ \text{Cov}(p_i, p_{i+j}) &= 0, \quad j \neq 0. \end{aligned}$$

We note that the variance of the inflation process is a linear combination of the variances of the three processes $\{u_i\}_{i=0,1,\dots}$, $\{v_i\}_{i=0,1,\dots}$ and $\{z_i\}_{i=0,1,\dots}$.

⁵If one takes the view that this third source of randomness is not advisable to include, one can assume that its value is always 0 with probability 1, i.e. assuming a Dirac-distribution centered on zero for it. As one notices in the following developments, this third source of randomness is mainly used in the calibration phase in order to have an additional degree of freedom and has no impact on the theoretical development of the model.

Finally, we calculate the centered third and fourth moments: these may be needed in order to analyse their distribution in a more complete fashion:

$$\mathbb{E} \left[(p_i - \mathbb{E}(p_i))^3 \right] = (K_2)^3 \sigma^3 \text{Skew}(u_i) - (K_2)^3 \text{Skew}(v_i) + \text{Skew}(z_i) \quad (14)$$

$$\begin{aligned} \mathbb{E} \left[(p_i - \mathbb{E}(p_i))^4 \right] &= (K_2)^4 \sigma^4 \text{Kurt}(u_i) + (K_2)^4 \text{Kurt}(v_i) + \text{Kurt}(z_i) + \\ &+ 6(K_2)^4 \sigma^2 \text{Var}(u_i) \text{Var}(v_i) + 6(K_2)^2 \text{Var}(v_i) \text{Var}(z_i) + 6(K_2)^2 \sigma^2 \text{Var}(u_i) \text{Var}(z_i). \end{aligned} \quad (15)$$

3.1.3 Statistical properties of the short term nominal interest rate

The nominal interest rate n_i is defined as

$$n_i = \bar{n}(1 + \hat{n}_i) \quad (16)$$

where \bar{n} is the equilibrium nominal interest rate, which is the short rate that would be chosen by the central bank if the adjustment required by the Taylor rule was zero—as \hat{n}_i follows (8). This follows by the definition of \hat{n} as the log-linearised difference between the actual rate and equilibrium rate.

We take the equilibrium nominal rate \bar{n} as a constant input that can be obtained from research and is therefore not calibrated to any traded asset. We assume that the short rate is used to discount payments between different counterparties, i.e. it plays the role of the Libor rate (we are operating in a pre-Lehman environment): this assumption, albeit strong, simplifies the problem considerably.⁶

If we plug the Taylor rule (8) into (16) we rewrite the nominal rate as

$$n_{i+1} = \bar{n}(1 + \delta_x x_i + \delta_\pi p_i + v_i). \quad (17)$$

We can compact the notation by introducing the vectors

$$\delta = \begin{bmatrix} \delta_x \\ \delta_\pi \end{bmatrix}; \xi_i = \begin{bmatrix} x_i \\ p_i \end{bmatrix}.$$

Therefore the interest rate becomes $n_{i+1} = \bar{n}(1 + \delta^T \xi_i)$ where $(x)^T$ is the transpose of the vector x . The source of randomness v_i is now included in the dynamics of ξ_i , as can be seen from (10). Finally, by making use of (10) and (13) we get:

$$n_{i+1} = \bar{n}(1 + \delta^T A \mathbb{E}_i \xi_{i+1} + \delta^T K w_i + \delta^T e_2 z_i),$$

where $e_2^T = [0 \ 1]$.

By making use of this expression we calculate the mean, variance and the autocovariance of the nominal interest rate:

$$\begin{aligned} \mathbb{E}[n_{i+1}] &= \bar{n}(1 + \delta^T A \mathbb{E} \xi_{i+1}) \\ \text{Var}(n_{i+1}) &= (\bar{n})^2 (\delta^T K)^2 \sigma^2 \text{Var}(u_i) + (\bar{n})^2 (\delta^T K)^2 \text{Var}(v_i) + (\bar{n})^2 \delta_\pi^2 \text{Var}(z_i) \\ \text{Cov}(n_i, n_{i+j}) &= 0, j \neq 0. \end{aligned}$$

We note that the variance of the interest rate process is a linear combination of the variances of the three processes $\{u_i\}_{i=0,1,\dots}$, $\{v_i\}_{i=0,1,\dots}$ and $\{z_i\}_{i=0,1,\dots}$. We take the equilibrium rates, output gap and inflation expectations as inputs: they may be provided by macroeconomic research or can just be expression of the trader's views.

Similarly to what could be done for the inflation rate, we can also calculate the centered third and fourth

⁶To see this, we recall that in continuous time the short rate $n(t) = f(t, t) = \lim_{\Delta T \rightarrow 0} F(t, t, t + \Delta T)$ where the forward rate is defined as $F(t, S, T) = (P(t, S)/P(t, T) - 1)/(T - S)$ with $T > S$. In discrete time we define $n_i = f_{i,i} = F(t_i, t_i, t_{i+1})$, therefore getting $n_i = (1/P(t_i, t_{i+1}) - 1)/\tau_{i+1}$. As a consequence, the short rate can be used as the Libor rate, provided that there are no credit concerns in the interbank markets.

moments: these are needed in order to analyse their distribution in a more complete fashion:

$$\mathbb{E} \left[(n_i - \mathbb{E}(n_i))^3 \right] = [(\delta^T K)^3 \sigma^3 \text{Skew}(u_i) - (\delta^T K)^3 \text{Skew}(v_i) + (\delta_\pi)^3 \text{Skew}(z_i)](\bar{n})^3 \quad (18)$$

$$\begin{aligned} \mathbb{E} \left[(n_i - \mathbb{E}(n_i))^4 \right] &= [(\delta^T K)^4 \sigma^4 \text{Kurt}(u_i) + (\delta^T K)^4 \text{Kurt}(v_i) + (\delta_\pi)^4 \text{Kurt}(z_i) + \\ &+ 6(\delta^T K)^4 \sigma^2 \text{Var}(u_i) \text{Var}(v_i) + 6(\delta^T K \delta_\pi)^2 \text{Var}(v_i) \text{Var}(z_i) + 6(\delta^T K)^2 (\sigma \delta_\pi)^2 \text{Var}(u_i) \text{Var}(z_i)](\bar{n})^4. \end{aligned} \quad (19)$$

Finally, we calculate the covariance between the nominal rate n_{i+1} and the inflation p_i , both \mathcal{F}_i -measurable:

$$\text{Cov}(p_i, n_{i+1}) = \bar{n} K_2 (\delta^T K) \sigma^2 \text{Var}(u_i) + (\bar{n} K_2 (\delta^T K)) \text{Var}(v_i) + \bar{n} \delta_\pi \text{Var}(z_i). \quad (20)$$

The covariance depends on the Taylor rule parameters vector δ , which makes explicit the philosophy of our modelling approach: any dependence between the nominal interest rate and inflation is not specified exogenously but is a consequence of the central bank reaction function. Furthermore, the correlation becomes one if there is no uncertainty in the Taylor rule, i.e. $\text{Var}(v_i) = 0$ and $\text{Var}(z_i) = 0$: in this case the central bank reacts deterministically to any change in the economy.

The other interesting limit case is when the output gap evolves deterministically, i.e. $\text{Var}(u_i) = 0$, the central bank does not take any action, i.e. $\delta = 0$ and $\text{Var}(z_i) = 0$: rates evolve stochastically and correlation becomes -1 . As rates increase, the output gap decreases deterministically (because of the demand curve (7)), bringing down the inflation according to the Phillips curve (6). In this case the only source of randomness is the uncertainty in the short rate evolution.

The DSGE model augmented with the Taylor rule allows for this correlation to take values between -1 and 1 , depending on the central bank reaction function and the specification of the sources of randomness: this can be arguably regarded as an interesting feature of the model, because it does not impose any constraint on the correlation range.

3.1.4 Calibrating to rates and inflation smiles: the normal case

The prices of nominal rates and inflation caps/floors across different strikes and maturities are available from brokers or investment banks (for example the Bloomberg pages VOLS or RILO): we can thus deduce the caplet/floorlet prices. Unlike options on other underlyings, inflation options are quoted in prices, not in implied volatilities. By making some distributional assumptions on the nominal rates and inflation, we summarise the distribution using only a few parameters.

For example, we can assume a normal distribution and fit its volatility to the option prices for each maturity: this assumption is both convenient from an analytical perspective (closed formulas for option prices are obtained) and from a practitioner point of view: if rates are normally (and not lognormally) distributed, the distribution of their relative increments is skewed, (and not Gaussian as in the Black model). In this case we calibrate the variances of $\{u_i\}_{i=0,1,\dots}$, $\{v_i\}_{i=0,1,\dots}$ and $\{z_i\}_{i=0,1,\dots}$ to obtain the market implied variances for nominal rates and inflation. We calculate the market implied variances for $\{u_i\}_{i=0,1,\dots}$, $\{v_i\}_{i=0,1,\dots}$, and $\{z_i\}_{i=0,1,\dots}$ given the market implied variances of rates/inflation caplets/floorlets: a word of caution should be issued, as there is no guarantee to obtain positive variances from this basic algorithm. Negative variances could be floored to zero or more sophisticated root-searching algorithms can be used.

3.1.5 Measure change under normality assumptions

At this stage we make the measure change process $\{\mu_i\}_{i=0,1,\dots}$ explicit to use the real-world expectations to price derivatives in the risk-neutral measure. We define the measure change processes as discretely sampled exponential Gaussian martingale: this strategy allows one to obtain a positive martingale. A general introduction to exponential Lévy martingales can be found in Appelbaum [2].

To simplify the notation, we rewrite equation 2.3 including the variable z_i in matrix format:

$$\xi_i = A \mathbb{E}_i \xi_{i+1} + K w_i + e_2 z_i = A \mathbb{E}_i \xi_{i+1} + K \sigma u_i - K v_i + e_2 z_i. \quad (21)$$

Defining the matrix C as follows:

$$C = \begin{bmatrix} \sigma K_1 & -K_1 & 0 \\ \sigma K_2 & -K_2 & 1 \end{bmatrix}$$

and compacting all three sources of randomness in the three-dimensional vector ε_i defined as $\varepsilon_i = [u_i \quad v_i \quad z_i]$, the notation is further simplified into $\xi_i = A\mathbb{E}_i\xi_{i+1} + C\varepsilon_i^T$.

One notes that the variance-covariance matrix for the vector is written as:

$$\Sigma_i^\varepsilon = \begin{bmatrix} \text{Var}(u_i) & 0 & 0 \\ 0 & \text{Var}(v_i) & 0 \\ 0 & 0 & \text{Var}(z_i) \end{bmatrix} = \begin{bmatrix} \text{Var}(\varepsilon_i^1) & 0 & 0 \\ 0 & \text{Var}(\varepsilon_i^2) & 0 \\ 0 & 0 & \text{Var}(\varepsilon_i^3) \end{bmatrix}.$$

At this point we introduce the three-dimensional vector process λ_i defined as:

$$\lambda_i = \begin{bmatrix} \lambda_i^u \\ \lambda_i^v \\ \lambda_i^z \end{bmatrix}.$$

The quantities defined above are used to specify the measure change process $\{\mu_i\}_{i=0,1,\dots}$, following and generalising Shreve [38]. The measure change process is therefore defined as a multivariate Gaussian exponential martingale in the form: $\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_i} = \mu_i = e^{-\varepsilon_i \cdot \lambda_i - 1/2 \lambda_i^T \Sigma_i^\varepsilon \lambda_i}$.

One requires $\mu_0 = 1$ and the market price of risk vector process $\{\lambda_i\}_{i=0,1,\dots}$ to be regular enough for the measure change process $\{\mu_i\}_{i=0,1,\dots}$ to be a positive martingale (i.e. its expectation has always to be finite) and square-integrable. Moving to the risk-neutral measure \mathbb{Q} , one obtains that the new process $\nu_i = \varepsilon_i + \lambda_i$ is a zero-mean Gaussian process with independent realisations under \mathbb{Q} .

We rewrite the expression for the macroeconomic variables (output gap and inflation) once the measure change from \mathbb{P} to \mathbb{Q} has been performed: $\xi_i = A\mathbb{E}_i\xi_{i+1} + C\nu_i = A\mathbb{E}_i\xi_{i+1} + C\lambda_i + C\varepsilon_i^T$.

Informally, one can think to the linear function of the market prices of risk λ_i as a “wedge” that is premultiplied by some coefficients in the matrix C and then added to the deterministic linear function of the expectations $A\mathbb{E}_i\xi_{i+1}$ in order to calibrate the model to the traded prices of nominal bonds and inflation breakevens (through the relationship between nominal bonds, real bonds and inflation zero-coupon swaps).

Finally, one finds a compact expression for the nominal short rate n_i and the inflation rate p_i under \mathbb{Q} :

$$n_{i+1} = \bar{n}(1 + \delta^T \xi_i) = \bar{n}(1 + \delta^T (A\mathbb{E}_i\xi_{i+1} + C\nu_i^T)) = \bar{n}(1 + \delta^T (A\mathbb{E}_i\xi_{i+1} + C\lambda_i + C\varepsilon_i^T))$$

$$\begin{aligned} p_i &= A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} + K_2 w_i + z_i = A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} + \sigma K_2 u_i - K_2 v_i + z_i = \\ &= A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} + \sigma K_2 (u_i + \lambda_i^u) - K_2 (v_i + \lambda_i^v) + (z_i + \lambda_i^z) = A_{2,1}\mathbb{E}_i x_{i+1} + A_{2,2}\mathbb{E}_i p_{i+1} + h\nu_i^T \end{aligned}$$

where the vector h has been defined as $h = [\sigma K_2 \quad -K_2 \quad 1]$.

3.1.6 Calibrating to the nominal term structure

We show how to calibrate the model to the nominal interest rates observed in the market by making some approximations. We use market prices of one period discount factors to provide some expressions to be used in the calibration. We write:

$$P(t_0, t_i, t_{i+1}) = \mathbb{E}_0^\mathbb{P} \left[\frac{\psi_{i+1}}{\psi_i} \right] = \mathbb{E}_0^\mathbb{Q} [(1 + n_{i+1}\tau_{i+1})^{-1}] \cong \mathbb{E}_0^\mathbb{Q} [e^{-n_{i+1}\tau_{i+1}}]$$

where the last linearisation creates some error that can be reduced by calibrating the model on a finer time grid. The term τ_{i+1} is the year fraction: $\tau_{i+1} = t_{i+1} - t_i$.

The following step is to introduce the closed form expression for the nominal rate n_i :

$$\mathbb{E}_0^\mathbb{Q} [e^{-n_{i+1}\tau_{i+1}}] = \mathbb{E}_0^\mathbb{Q} \left[e^{-\bar{n}(1+\delta^T(A\mathbb{E}_i\xi_{i+1}+C\nu_i^T))\tau_{i+1}} \right] = e^{-\bar{n}\tau_{i+1}(1+\delta^T(A\mathbb{E}_i\xi_{i+1})) + \frac{\bar{n}^2\tau_{i+1}^2}{2}\delta^T C \Sigma_i^\varepsilon C^T \delta}$$

By taking the expectations under the normality assumption for the vector ν_i , the one-period forward discount factors approximated closed form is:

$$P(t_0, t_i, t_{i+1}) \cong e^{c_1 + c_2 \text{Var}(u_i) + c_3 \text{Var}(v_i) + c_4 \text{Var}(z_i)} \quad (22)$$

where:

$$c_1 = -\tau_{i+1} \bar{n} (1 + \delta^T (A \mathbb{E}_i \xi_{i+1}))$$

$$c_2 = \frac{1}{2} (\tau_{i+1} \bar{n} \delta^T K \sigma)^2$$

$$c_3 = +\frac{1}{2} (\tau_{i+1} \bar{n} \delta^T K)^2$$

$$c_4 = \frac{1}{2} (\tau_{i+1} \bar{n} \delta_\pi)^2.$$

3.1.7 Calibrating to the ZCIIS

As shown in Brigo & Mercurio [7], the value of a zero-coupon inflation index swap (ZCIIS) can be regarded as the difference between the real and nominal zero-coupon bond prices with the same maturity date. For the full definition of the real bond and term structure we refer to Hughston [23].

We exploit the model-independent relationship between real and nominal bond to write:

$$P^R(t_0, t_{i+1}) = P(t_0, t_{i+1}) + ZCIIS(t_0, t_{i+1}).$$

Since we observe the market prices of nominal bonds and ZCIIS for different maturities, we deduce the value of a real bond, even if these instruments are not traded in the market.

We assume that the real bond pays at maturity t_{i+1} the unit nominal multiplied by the underlying inflation index appreciation between times t_0 and t_i : this is to introduce the inflation publication lag in the formula, which becomes necessary since in reality the inflation rate is only published after a time lag.

The approximated closed form is obtained as follows:

$$P^R(t_0, t_{i+1}) = \mathbb{E}_0^{\mathbb{P}} \left[\frac{I_i \psi_{i+1}}{I_0 \psi_0} \right] = \mathbb{E}^{\mathbb{P}} \left[\frac{I_i}{I_0} \psi_{i+1} \right] = \mathbb{E}^{\mathbb{Q}} \left[\frac{I_i}{I_0} \prod_{j=1}^{i+1} \frac{1}{1 + \tau_j n_j} \right] = \mathbb{E}^{\mathbb{Q}} \left[\prod_{j=1}^{i+1} \frac{1 + \tau_{j-1} p_{j-1}}{1 + \tau_j n_j} \right]$$

By making some straightforward Taylor expansions the last expression can be rewritten as:

$$P^R(t_0, t_{i+1}) = \mathbb{E}^{\mathbb{Q}} \left[\prod_{j=1}^{i+1} \frac{e^{\log(1 + \tau_{j-1} p_{j-1})}}{e^{\log(1 + \tau_j n_j)}} \right] \cong \mathbb{E}^{\mathbb{Q}} \left[\prod_{j=1}^{i+1} \frac{e^{\tau_{j-1} p_{j-1}}}{e^{\tau_j n_j}} \right] = \mathbb{E}^{\mathbb{Q}} \left[e^{\sum_{j=1}^{i+1} (\tau_{j-1} p_{j-1} - \tau_j n_j)} \right].$$

We assume that $p_0 = 0$ and focus the attention on the one-period forward real discount factor.

By the same Gaussianity assumptions used above, the following closed formula is obtained by plugging (13) into the above expression:

$$\begin{aligned} P^R(t_0, t_i, t_{i+1}) &\cong \mathbb{E}_0^{\mathbb{Q}} [e^{-\tau_{i+1} n_{i+1} + \tau_i p_i}] = \mathbb{E}_0^{\mathbb{Q}} [e^{-\tau_{i+1} \bar{n} (1 + \delta^T (A \mathbb{E}_i \xi_{i+1} + C \nu_i^T)) + \tau_i (A_{2,1} \mathbb{E}_i x_{i+1} + A_{2,2} \mathbb{E}_i p_{i+1} + h \nu_i^T)}] = \\ &= \mathbb{E}_0^{\mathbb{Q}} [e^{-\tau_{i+1} \bar{n} (1 + \delta^T A \mathbb{E}_i \xi_{i+1}) + \tau_i (A_{2,1} \mathbb{E}_i x_{i+1} + A_{2,2} \mathbb{E}_i p_{i+1}) + \nu_i^T (\tau_i h - \tau_{i+1} \bar{n} \delta^T C)}] = e^{b_1 + b_2 \text{Var}(u_i) + b_3 \text{Var}(v_i) + b_4 \text{Var}(z_i)} \quad (23) \end{aligned}$$

where

$$b_1 = \tau_i A_{2,1} \mathbb{E}_i x_{i+1} + \tau_i A_{2,2} \mathbb{E}_i p_{i+1} - \tau_{i+1} \bar{n} (1 + \delta^T A \mathbb{E}_i \xi_{i+1})$$

$$b_2 = \frac{1}{2} (\tau_i K_2 \sigma - \tau_{i+1} \bar{n} \delta^T K \sigma)^2$$

$$b_3 = \frac{1}{2} (\tau_i K_2 - \tau_{i+1} \bar{n} (\delta^T K))^2$$

$$b_4 = \frac{1}{2} (\tau_i - \tau_{i+1} \bar{n} \delta_\pi)^2.$$

We stress that the variances calibrated from option prices are taken as input in the above expression. The two approximated closed formulas for the nominal (22) and the real bond (23) can be used to find the values of λ_i that calibrate the model to the market, given the variances of the distributions of the shock factors u_i , v_i , and z_i .

To conclude this section, we observe that the adaptation of the DSGE model to pricing proposed above is extremely respectful of the the original macroeconomic model, but for this reason is also not straightforward to price derivatives. In fact, to obtain closed forms for the nominal and real bonds one has to resort to approximations and linearisations of exponentials, which are doable but not elegant: the model offers an insight of the macroeconomic forces operating behind the yield curve and the inflation dynamics, but all pricing has to happen using Monte Carlo simulations, which can be cumbersome and time consuming. Interestingly, the above section shows a first attempt to bridge the gap between two disciplines (monetary macroeconomics and financial mathematics) that are dealing with the same problem (inflation) in two different ways (DSGE modelling versus arbitrage pricing): this represent a step forward in the same direction indicated by Hughston & Macrina [25], who derive some inflation dynamics from a macroeconomic model — even if there is no concept of central bank policy.

With these ideas in mind, in the following section we propose some dynamics that, while retaining the most significant aspects of the DSGE model presented in the previous sections, are more tractable from a derivatives pricing perspective. It is important to stress that the new dynamics we propose are not a one-to-one translation of the discrete-time DSGE model, but rather they are inspired by it and take into account that in the post-Lehman environment the short rate is not the only policy tool used by the central bank.⁷ To ensure that the proposed dynamics are meaningful, we bring some empirical evidence that shows that the proposed dynamics are realistic: finally, we show that the discrete-time DSGE model and the continuous-time model proposed generate similar distributions for the main economic variables.

4 Building the continuous-time version

Here we propose a strategy to loosely translate the DSGE model into continuous time by making some assumptions. Therefore we show that some continuous-time dynamics can be derived from a widely accepted macroeconomic model: they are used in the next section to build the inflation model. From this point, the notation for the variable y in continuous time will be $y(t)$.

The following assumptions are made:

1. There is no price flexibility for the firms, i.e. $\omega = 1$ and $k = 0$. This assumption is reasonable as markets tend to be far from the perfect competition model, and therefore prices are sticky, especially over a shorter time step.
2. The one-period subjective discount factor is equal to the inverse of the inflation targeting parameter: $\beta \delta_\pi = 1$. This assumption is sensible because, when the central bank fights inflation aggressively (i.e. $\delta_\pi \gg 1$), interest rates will increase, pushing down the discount factor β .
3. The GDP growth rate is modelled in the same way as the output gap. In fact, because the output gap is defined as the difference between the actual and the potential GDP growth rate, and because

⁷When short rates are low or ineffective to stimulate the economy, the central bank can purchase assets to reduce long term interest rates and increase the money supply.

the latter is an abstract concept (normally deemed to be constant over time), this means adding the constant potential growth rate to the output gap.

4. The GDP growth rate is defined as the percentage change of the GDP level from one period to the next one: $x_i = (X_i - X_{i-1})/X_{i-1}$. One can change the notation and write: $x_{t_i} = (X_{t_i} - X_{t_{i-1}})/X_{t_{i-1}}$. Furthermore, one can generalise the time step and write: $\Delta t_i = t_i - t_{i-1}$, therefore obtaining: $x_{t_i} = (X_{t_i} - X_{t_{i-\Delta t_i}})/X_{t_{i-\Delta t_i}} = \Delta X_{t_i}/X_{t_{i-\Delta t_i}}$. Moving to continuous time one can write $x(t)$ as $dX(t)/X(t)$. One needs to assume that the positive process $\{X_i\}_{i=0,1,\dots}$ is regular enough for the limit to exist.
5. A similar line of thought can be followed to see how one moves from the discrete-time definition of inflation, as the percentage change in the price index level ($p_i = (I_i - I_{i-1})/I_{i-1}$), to the equivalent continuous-time definition ($p(t)$ is written as $dI(t)/I(t)$). Again we make an obvious request of positivity for the price index process $\{I_i\}_{i=0,1,\dots}$.
6. There are measurement errors and other sources of uncertainty for both inflation and growth rate, modelled by the m -dimensional zero mean random variable z_i . The m components of this random variable (called z_i^j , with $1, 2, \dots, j, \dots, m$) are independent from each other. The random variable z_i is also independent from w_i . The effects of the shock z_i^j on x_i and p_i are modelled by the m -dimensional real-valued processes $\{a_i\}_{i=0,1,\dots}$ and $\{b_i\}_{i=0,1,\dots}$, where their single components have notation $a_{i,j}$ and $b_{i,j}$.
7. The product of the expectations terms by some constants that appear in the DSGE model can be written as $\sigma/(\sigma + \delta_x + k\delta_\pi)\mathbb{E}_i x_{i+1} = m_X(t_i)(t_{i+1} - t_i)$ and $(k + \beta(\sigma + \delta_x))/(\sigma + \delta_x + k\delta_\pi)\mathbb{E}_i p_{i+1} = m_I(t_i)(t_{i+1} - t_i)$ respectively. We assume that the quantities $m_X(t_i)$ and $m_I(t_i)$ are realisations of adapted stochastic processes. This means that these expectations are not dependent on the chosen time lag, and can be written as the product by a real function of time ($m_X(t_i)$ and $m_I(t_i)$ respectively) and the chosen time lag. One can generalise the time lag by writing $\sigma/(\sigma + \delta_x + k\delta_\pi)\mathbb{E}_{t_i} x_{t_i + \Delta t_i} = m_X(t_i)\Delta t_i$ and $(k + \beta(\sigma + \delta_x))/(\sigma + \delta_x + k\delta_\pi)\mathbb{E}_{t_i} p_{t_i + \Delta t_i} = m_I(t_i)\Delta t_i$ respectively. When one moves to continuous time, $\Delta t_i \rightarrow dt$, and the real quantities $m_X(t)$ and $m_I(t)$ do not change. Therefore one can write the products of expectation terms and constants as a continuous time drift ($m_X(t)dt$ and $m_I(t)dt$ respectively).
8. The random variables u_i and z_i^j are independent and normally distributed, with mean 0 and unit variance.
9. The random variables u_i and z_i^j are independent from their previous levels. For example, taken u_i , one can write $\text{Cov}(u_i, u_l) = \delta_{i,l}$. In this context $\delta_{i,l}$ is the Kronecker's delta sign, taking value 0 in all cases where $i \neq l$ and 1 when $i = l$.
10. Taken the random variable w_i , one can introduce the random variable U_i , defined as $U_i = \sum_{k=1}^i w_k$, with $U_0 = 0$. Based on all the assumptions made, one can show that $U_i \sim N(0, i)$. By construction, the process $\{U_i\}_{i=0,1,\dots}$ will have zero mean, independent increments and $U_i - U_l \sim N(0, i - l)$. The increment $U_{i+k} - U_{l+k}$ has the same distribution as the increment $U_i - U_l$, for each k .
11. By generalising the time lag, one sees that $\Delta U_{t_i} = U_{t_i} - U_{t_i - \Delta t_i} \sim N(0, \Delta t_i)$. Moving to continuous time one gets a Brownian motion. A similar discussion can be held for the M -dimensional random variable z_i , which becomes an m -dimensional Brownian motion with independent components. Shreve [38] gives full details of this procedure to build the Brownian motion starting from a discrete-time Gaussian process.
12. To compact notation, one introduces the $m+1$ -dimensional (or alternatively n -dimensional) vectors, defined as $s_{t_i}^X = [\sigma, a_{t_i}^1, \dots, a_{t_i}^M]$, and $s_{t_i}^I = [0, b_{t_i}^1, \dots, b_{t_i}^M]$. The idea is to compact all the random terms to express them using a lighter notation.
13. To move to continuous time, one assumes that the processes $\{s_{t_i}^X\}_{t_i=0,1,\dots}$ and $\{s_{t_i}^I\}_{t_i=0,1,\dots}$ are regular enough for the limits $s_{t_i}^X \rightarrow s_X(t)$ and $s_{t_i}^I \rightarrow s_I(t)$ to exist.

The system 2.3 can be rewritten in discrete time using a generic time step Δt_i as:

$$\begin{bmatrix} x_{t_i} \\ p_{t_i} \end{bmatrix} = \begin{bmatrix} m_X(t_i) \\ m_I(t_i) \end{bmatrix} \Delta t_i + \begin{bmatrix} \sigma \\ 0 \end{bmatrix} (w_{t_i}) \Delta t_i^{1/2} + \sum_{j=1}^M \begin{bmatrix} a_{i,j} \\ b_{i,j} \end{bmatrix} (z_{t_i}^j) \Delta t_i^{1/2} \quad (24)$$

From the assumptions made above, the two above equations can be translated in continuous time as follows:

$$dX(t)/X(t) = m_X(t)dt + s_X(t) \cdot dW(t) \quad (25)$$

$$dI(t)/I(t) = m_I(t)dt + s_I(t) \cdot dW(t), \quad (26)$$

where $\{W(t)\}_{t \geq 0}$ is an n -dimensional Brownian motion. The notation \cdot is used to refer to the vector product. At this stage one can complement this model with some dynamics for the expectations of the drift: in fact, as shown in the following section, empirical evidence suggests that expectations themselves are subject to frequent revisions (as the economic agents process new information and data) and therefore are themselves stochastic. A possible expression for the dynamics of the expectations is the following:

$$dm_X(t) = a_X(t)dt + b_X(t) \cdot dW(t) \quad (27)$$

$$dm_I(t) = a_I(t)dt + b_I(t) \cdot dW(t). \quad (28)$$

where the processes $\{a_X(t)\}_{t \geq 0}$, $\{a_I(t)\}_{t \geq 0}$, $\{b_X(t)\}_{t \geq 0}$ and $\{b_I(t)\}_{t \geq 0}$ are deterministic processes regular enough for the SDEs to be integrated and to have a unique strong solution. To conclude, the above stochastic differential equations are derived from a well-established macroeconomic model. They are consistent with empirical evidence (as shown in the next section) and will be used in the following section as a part of a wider setup to build a structural continuous-time pricing model for inflation derivatives, based on macroeconomic assumptions.

4.1 Testing the dynamics against empirical evidence

In this section we show some economic time series to confirm that, over time, the growth rate of real GDP and of the price index are stationary processes that show some randomness. The actual levels of real GDP and price index are growing in an exponential fashion over time: these two observations confirm that the choice of a Brownian motion with time-changing coefficients and stochastic drift is sensible.

Evidence for the below facts can be shown for the US and the UK economy, however similar results hold for all economies.

Fact 1 - Over time both price indexes and GDP have grown steadily, as shown by the first four figures.

Fact 2 - Over time their growth rate has been subject to some randomness, as shown by the fifth to the eighth figure of this section.

Further, we show some evidence of expectations (or forecast) of UK GDP growth rate (compiled by Bloomberg) and of the US inflation rate (compiled by the University of Michigan): both series show that the expectations themselves are stochastic, which suggests that the assumption of assuming the processes for the expectations is sensible and consistent with empirical evidence.

Fact 3 - Growth rate and inflation expectations are subject to randomness: this is shown by the last two figures of this section.

4.2 Comparing the DSGE model with the continuous-time model

This section shows that the discrete-time DSGE model and the continuous-time model we propose can deliver similar distributions for the main economic variables if one parametrizes them in a consistent way. Therefore in the following section we will choose the continuous-time model to develop the theory as it is superior compared to the DSGE model as far as its analytical tractability is concerned.

Further, in the following section we will show that one can find closed form expressions in the continuous-time model to both the nominal and inflation term structure, to both nominal rates and inflation options, and to year-on-year inflation forward, without having to resort to the linearisations and approximations seen earlier in this section when dealing with the discrete-time DSGE model.

In order to obtain similar distributions for the most relevant financial quantities, one applies a moment-matching technique across different models. We assume that all parameters in the continuous-time model are expressed as right-continuous step functions and that the dimensionality of the Brownian motion is 3. We focus our attention on second order moments, as the first order moments are straightforward to match.

Inflation rate. In the discrete-time DSGE model, we showed that the variance of the inflation rate is:

$$\text{Var}(p_i) = (K_2)^2(\sigma^2 \text{Var}(u_i) + \text{Var}(v_i)) + \text{Var}(z_i).$$

In the next section (see 80 on page 37) we show that the diffusion term of the inflation rate (approximated by the ratio $dI(t)/I(t)$) is $[b_I(t)(T-t) + s_I(t)]$

This implies that the total variance over the first year ($t = 0$ and $T = 1$) is $\sum_{i=1}^3 [b_I(0) + s_I(0)]^2$.

Therefore it makes sense to match the two conditions by requesting that:

$$\text{Var}(p_i)/(t_i - t_{i-1}) = [(K_2)^2(\sigma^2 \text{Var}(u_i) + \text{Var}(v_i)) + \text{Var}(z_i)]/(t_i - t_{i-1}) = \sum_{i=1}^3 [b_I(0) + s_I(0)]^2 \quad (29)$$

Short rate. A similar method can be applied to the variance of the nominal short rate, that in the DSGE set-up is calculated as:

$$\text{Var}(n_{i+1}) = (\bar{n})^2(\delta^T K)^2 \sigma^2 \text{Var}(u_i) + (\bar{n})^2(-\delta^T K)^2 \text{Var}(v_i) + (\bar{n})^2 \delta_\pi^2 \text{Var}(z_i).$$

Because the short term nominal rate level at time t_{i+1} is independent from its level at the previous time t_i (this follows because the nominal rate is a linear combination of the output gap and inflation, both of which are driven by Gaussian processes that are independent from their own realisations over time), one can write the variance of the change in the nominal rate as:

$$\begin{aligned} \text{Var}(n_{i+1} - n_i) &= \text{Var}(n_{i+1}) + \text{Var}(n_i) = \\ &= (\bar{n})^2[(\delta^T K)^2 \sigma^2 (\text{Var}(u_i) + \text{Var}(u_{i-1})) + (-\delta^T K)^2 (\text{Var}(v_i) + \text{Var}(v_{i-1})) + \delta_\pi^2 (\text{Var}(z_i) + \text{Var}(z_{i-1}))]. \end{aligned}$$

In the next section (see 59 on page 31) we will show that the diffusion term of the nominal short rate differential $dn(t)$ is:

$$-\frac{h_x b_X(t) + h_p b_I(t)}{\zeta(t)}$$

The matching condition is therefore:

$$\begin{aligned} [(\bar{n})^2[(\delta^T K)^2 \sigma^2 (\text{Var}(u_i) + \text{Var}(u_{i-1})) + (-\delta^T K)^2 (\text{Var}(v_i) + \text{Var}(v_{i-1})) + \delta_\pi^2 (\text{Var}(z_i) + \text{Var}(z_{i-1}))]]/(t_i - t_{i-1}) &= \\ = -\frac{h_x b_X(t) + h_p b_I(t)}{\zeta(t)} \end{aligned} \quad (30)$$

For the moment we assume that $\zeta(t)$ is a mere positive calibration function, and that the real positive parameters h_x and h_p are taken exogenously. In fact, as it will be shown in the next section, they have a precise financial meaning. This said, the purpose of this exercise at this stage is simply to show that some statistical properties in two different models can be matched.

Covariance between nominal short rate and inflation. The covariance in the DSGE model is:

$$\text{Cov}(p_i, n_{i+1}) = \bar{n} K_2 (\delta^T K) \sigma^2 \text{Var}(u_i) + (\bar{n} K_2 (\delta^T K)) \text{Var}(v_i) + \bar{n} \delta_\pi \text{Var}(z_i).$$

Because the short term nominal rate level at time t_{i+1} is independent from its level at the previous time t_i (as discussed above), this covariance can be interpreted also as

$$\text{Cov}(p_i, n_{i+1}) = \text{Cov}(p_i, n_{i+1} - n_i).$$

The correlation is calculated as follows:

$$\text{Corr}(p_i, n_{i+1} - n_i) = \frac{\bar{n}K_2(\delta^T K)\sigma^2\text{Var}(u_i) + (\bar{n}K_2(\delta^T K))\text{Var}(v_i) + \bar{n}\delta_\pi\text{Var}(z_i)}{(K_2)^2(\sigma^2\text{Var}(u_i) + \text{Var}(v_i)) + \text{Var}(z_i))^{1/2}(\text{Var}(n_{i+1}) + \text{Var}(n_i))^{1/2}}$$

By doing some basic calculations, and by taking into account results 59 on page 31 and 80 on page 37, one can calculate the instantaneous covariance of the nominal short rate change and the inflation rate between times t and T :

$$-[b_I(t)(T-t) + s_I(t)][\zeta(t)^{-1}(h_x b_X(t) + h_p b_I(t))]$$

Therefore the matching condition is:

$$\text{Cov}(p_i, n_{i+1} - n_i)/(t_i - t_{i-1}) = -[b_I(t)(T-t) + s_I(t)][\zeta(t)^{-1}(h_x b_X(t) + h_p b_I(t))]. \quad (31)$$

Example. To show the application of the above methodology, we simulate over the first year the GDP growth rate, the inflation rate, and the short nominal interest rate over 5,000 Monte Carlo trials. We assume that the shocks in the DSGE model are normally distributed with some variances that will be calibrated below.

The assumptions made on the economy and the financial market are the following:

1. Market agents are risk-neutral. This translates in a σ parameter of 0 in the DSGE model, and in zero market prices of risk, both in the discrete-time and in the continuous-time Gaussian processes.
2. The inflation rate is expected to be 3% in the first year, with a standard deviation of 1.1%. This standard deviation can be either an empirical estimate or can be inferred from traded derivatives markets. The source of this standard deviation is not relevant in this exercise.
3. The output gap is expected to be -2% in the first year.
4. The potential growth rate of the economy is 2%.
5. The equilibrium level of the short term nominal rate is 4%.
6. The standard deviation of the nominal short rate is 0.45%. The short rate is currently at 2.1%.
7. The central bank is attaching three times more importance to fighting inflation than to stimulating growth.
8. The correlation between the nominal rate change and the inflation rate is positive (given that the central bank is targeting inflation in a very aggressive way), and is 65%.

The time index at 1 means that the parameter is relative to the first year, that is the point in time that we are simulating. The parametrisation we choose for the DSGE model is the following:

Parameter	Value	Comment
σ	0	Agents are risk-neutral
k	0.01	Prices are sticky
δ_π	3	The central bank fights inflation aggressively
δ_x	1	The central bank is not targeting growth
β	0.95	Standard subjective discount factor
$\text{Var}(u_1)$	0.01	
$\text{Var}(v_1)$	0.01	
$\text{Var}(z_1)$	0.0001	

Moving to the continuous time model, we propose the following parametrisation based on the same economic assumptions seen above, applying the moment matching conditions stated above. The time indexes are either 0 (initial condition) or 1 (final condition). Between these two times one can think to the parametric functions like $a_X(t)$, $a_I(t)$, $b_X(t)$, $b_I(t)$, $s_I(t)$, and $s_X(t)$ as right-continuous step functions: because the dimensionality of the Brownian motion is 3, there are three values for the volatility vectors specified below. The Monte Carlo simulation has been run in one time step equal to one year. Finally, some of the model parameters that appear in the below table have not been explained in the continuous-time construction seen above, but will be introduced in the following section. The aim of this section is to show that the two models provide results that are broadly in line, however the full explanation of the continuous-time model will take place in the following section.

Parameter	Value	Value (2)	Value (3)	Comment
h_x	3			Please refer to next section for this parameter
h_p	1			Please refer to next section for this parameter
\bar{x}	2%			Please refer to next section for this parameter
\bar{p}	2%			Please refer to next section for this parameter
$\zeta(0)$	2.015			Please refer to next section for this parameter
$a_X(0)$	0.5%			
$a_I(0)$	-1.5%			
$\mu_X^p(0)$	-0.5%			
$\mu_I^p(0)$	4.5%			
$s_X(0)$	0.01	0.01	0.01	
$s_I(0)$	0.01	0.01	0.01	
$b_X(0)$	0.004	-0.01	-0.00005	
$b_I(0)$	-0.001	0.0005	-0.0005	

Results Here we show a table comparing target levels, results in the DSGE model, results in the continuous-time model for the short rate change and the inflation rate in the first year. Both the marginal and the joint distributions are matched well.

Statistic	Target	DSGE simulation	Continuous-time simulation
$\mathbb{E}[n_{i+1} - n_i]$	2%	2.01%	1.98%
$\mathbb{E}[p_i]$	3%	3.01%	2.97%
$\text{StDev}[n_{i+1} - n_i]$	0.45%	0.42%	0.44%
$\text{StDev}[p_i]$	1.1%	1.08%	1.07%
$\text{Corr}[n_{i+1} - n_i, p_i]$	65%	64.16%	69.07%

5 Inflation derivatives pricing with a central bank reaction function: the CTCB model

In this section we propose a continuous-time model to price inflation-linked derivatives by use of a model that explicitly takes into account the economic dynamics and the central bank behaviour. Therefore, as it happened in the discrete-time DSGE model analysed in the previous section, the co-movement of interest rates and inflation is not specified exogenously but rather is the result of central bank policy.

To achieve this, we make some standard assumptions regarding the structure of the financial market (absence of arbitrage) and model the relative changes of both real GDP (Gross Domestic Product) and

Price Index using Brownian motions with stochastic drifts. Furthermore the central bank trades nominal bonds to change the money supply in the economy, to keep the growth rate and inflation around some pre-specified targets (see for example Walsh [42]). These bond trades have an impact on the nominal bond prices, and therefore on the term structure of interest rates. Normally inflation-linked pricing models model the co-movement of inflation and nominal interest rates exogenously, without specifying the economic rationale behind this: we think that bridging the gap between economics and finance is beneficial for both disciplines.⁸

The advantages of this approach are manifold. Firstly, the dynamics assumed in this model appear to be consistent with the behaviour of central banks in recent years, when significant purchases of bonds (the so-called “quantitative easing”) have been made since short term interest rates have almost reached the zero lower bound in many developed economies. One can ask why it is important that a pricing model generates asset prices using realistic dynamics: after all, this could be irrelevant once the model has calibrated to a set of market observables. The answer is that such model would fail to minimise hedging profit and loss volatility if a dynamic hedging simulation was run and would make realistic stress testing difficult.⁹

Secondly, this approach does not rely on the so-called “Forex Analogy”, which assumes the existence of the “real” economy (see Hughston [23]). The Forex analogy has roots in the economic theory (see Fisher [20]). We will use this only as a theoretical tool, however the quantities we model are all market observables: this makes this model different from the Jarrow-Yildirim model (see Jarrow & Yildirim [29] or Brody, Crosby & Li [9]). The main advantage is that the model parameters are calibrated in a transparent way to liquid market observables (nominal bonds, inflation swaps, nominal and inflation caps and floors), as opposed to using and estimating a real rate volatility which is hardly observable in the market. In practice, one avoids taking costly reserves or valuation adjustments, as required by accounting principles, freeing up capital. Examples of models that do not rely on the Forex Analogy can be found in Dodgson & Kainth [18], in Mercurio [34] or in Brigo & Mercurio [7]: however all these models are not based on macroeconomic foundations.

Thirdly, although the model is extremely complex and takes into account many market features (dynamics for the price index and the real economy, their expectations, the no-arbitrage principle, central bank policy, liquidity effects), we show that the dynamics of the short term nominal rate can be reconciled with a well-established short interest rate model (the generalised Hull-White model, which is a time-varying parameters version of the Ornstein-Uhlenbeck process). This provides both an elegant link with the established theory and some closed forms useful for its calibration.

Fourthly, year-on-year forward closed forms are derived, showing that the model remains tractable even if it is based on realistic assumptions.

Fifthly, the extension of this model to the open economy is straightforward.

Sixthly, the calibration of this model is computationally not intensive, which allows fast pricing of all kind of trades, from inflation options, to year-on-year caps and floors, to more complex inflation structures as LPI (Limited Price Index). The main reason for this computational simplicity is that we propose a separable calibration. This point will be then fully developed in the following section.

The reader interested in the inflation derivatives market can refer to some marketing notes edited by investment banks, like Barclays [3] or Lehman Brothers [31], or alternatively Deacon, Derry & Mirfendereski [17], Benaben [5], Campbell, Shiller & Viceira [12], McGrath & Windle [33], and Jäckel & Bonneton [27].

⁸Another example of inflation-linked pricing based on sound economic assumptions can be found in Hughston & Macrina [24], Hughston & Macrina [25], and Alexander [1]: the spirit of these papers has been a source of inspiration for the current one, as they use a microeconomic approach based on Sidrauski [39] to determine the continuous-time dynamics of the price index (however they do not model the central bank reaction function).

⁹Dynamic hedging simulations are used to assess the quality of a model: the idea is to generate some “real world” dynamics and assess how the delta hedging done through a model performs, in terms of reducing hedging profit and loss volatility (“slippages”). Examples of these techniques can be found, for example, in Rebonato [35].

6 Inflation model assumptions

6.1 Probabilistic set-up

1. The model is set in continuous time t . Time is a positive real number and is expressed in years. From this point the notation for the continuous-time variable y at time t is $y(t)$.
2. Randomness is modelled via a n -dimensional \mathbb{P} -Brownian motion $\{W(t)\}_{t \geq 0}$. The probability measure \mathbb{P} is fully defined in the next point. The n components of the Brownian motion $W(t)$ are independent.
3. We work with the probability triplet $\{\Omega, \mathcal{F}, \mathbb{P}\}$ equipped with the natural filtration $\{\mathcal{F}_t\}_{t \geq 0}$ generated by the Brownian motion $\{W(t)\}_{t \geq 0}$. All filtration-related concepts are defined with respect to this filtration. In particular \mathbb{P} is the real-world (“physical”) probability measure. If no probability measure is specified, the expectation is taken with respect to the real-world measure (\mathbb{P}).
4. Derivatives pricing is carried out in the \mathbb{P} measure via the pricing kernel (defined below), or in the risk neutral measure \mathbb{Q} (defined as the pricing measure which is characterised by having the money market account $B(t)$ as numeraire), or finally in the T -forward measure \mathbb{Q}^T , defined as the pricing measure using the bond price $P(t, T)$ as numeraire. The bonds $P(t, T)$ are defined in detail in the next section. Expectations of a payoff Π taken under the generic measure \mathbb{M} given the information available at time t are denoted as $\mathbb{E}^{\mathbb{M}}[\Pi | \mathcal{F}_t]$ or alternatively $\mathbb{E}_t^{\mathbb{M}}[\Pi]$.

6.2 Financial instruments

All instruments listed below and their related quantities are modelled as $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted stochastic processes and are regular enough to ensure the existence of the expectations introduced later. The list of instruments is not exhaustive but only contains the ones needed to build the model.

1. **Nominal zero coupon bonds**, that pay with certainty one unit of currency at time T , have price $P(t, T)$ at time t . There exists a continuum of bond prices, i.e. $T \in [t, +\infty)$.¹⁰ From the nominal bond prices one can derive all kind of rates, for example instantaneous forward rates $f(t, T) = -\partial \log(P(t, T)) / \partial T$ and the short rate $n(t) = f(t, t)$. See Brigo & Mercurio [7] for further details.
2. **Money market account** $B(t)$, with dynamics $dB(t) = n(t)B(t)dt$, $B(0) = 1$.
3. **Price Index** $I(t)$, which is a positive stochastic process that reflects the price level of the economy. Its dynamics are specified later in the economy set-up.
4. **Zero-Coupon Inflation Index Swaps ZCIIIS**(t, T): the inflation leg pays $I(T)/I(t) - 1$, while the fixed leg pays $(1 + K(t, T))^{T-t} - 1$. Both payments happen at maturity T and there is no time lag. The inflation breakeven $K(t, T)$ is agreed at time t : there exists a continuum of inflation breakevens, i.e. $T \in [t, +\infty)$.¹¹ In a zero-coupon inflation swap the strike $K(t, T)$ is such that the expected value at maturity of the swap is zero: $\mathbb{E}_t^{\mathbb{Q}}[(I(T)/I(t) - (1 + K(t, T))^{T-t})B(t)/B(T)] = 0$.
5. **Inflation bonds** $P^I(t, T)$, which pay at time T the level of the price index $I(T)$, with no time lag. There exists a continuum of inflation bond prices, i.e. $T \in [t, +\infty)$. They are also known as “linkers”. Because we are working in a market without liquidity concerns, the inflation dynamics implied by the inflation bond prices are the same as the ones implied by the inflation swap market. The zero-coupon linker price is $P^I(t, T) = \mathbb{E}_t^{\mathbb{Q}}[I(T)B(t)/B(T)]$. Normally these bonds have an implicit deflation floor: the bond holder will not be paying the issuer in case of deflation.

¹⁰When calibrating the model to market observables, this assumption will be relaxed because only a finite amount of bond maturities are quoted on the market.

¹¹When calibrating the model to market observables, this assumption will be relaxed because only a finite amount of inflation swaps maturities are quoted on the market.

6.3 Financial market

The assumptions regarding the financial market are standard:

1. There is no credit risk in the economy.
2. The financial market is arbitrage-free. A thorough treatment of absence of arbitrage and its implications can be found in Björk [10], Cochrane [15], or Duffie [19].
3. Assuming that we use the money market account $B(t)$ as numeraire, we are working in the risk neutral measure \mathbb{Q} . This implies that the bond price \mathbb{Q} -dynamics are given by

$$dP(t, T)/P(t, T) = n(t)dt + \sigma_P(t, T) \cdot dW^{\mathbb{Q}}(t) \quad (32)$$

where the bond volatility $\sigma_P(t, T)$ is an n -dimensional deterministic process.¹² These volatilities will be referred to as “model” volatilities in the calibration section, as opposed to volatilities implied by market prices of options. The form of these bond volatilities is left general at this point, however at a later stage we will be able to fully characterise them in terms of model parameters.

4. The Radon-Nikodym derivative $L(t) = d\mathbb{Q}/d\mathbb{P}|_{\mathcal{F}_t}$ has the dynamics: $dL(t) = -L(t)\lambda(t) \cdot dW^{\mathbb{P}}(t)$, where $\lambda(t)$ is an n -dimensional deterministic process. The process $\{\lambda(t)\}_{t \geq 0}$ is called “market price of risk”.
5. The pricing kernel $\psi(t)$ defined as $\psi(t) = L(t)/B(t)$ has dynamics: $d\psi(t)/\psi(t) = -n(t)dt - \lambda(t) \cdot dW^{\mathbb{P}}(t)$. The pricing kernel has many useful properties, however for these purposes we remember that $P(t, T) = \mathbb{E}_t^{\mathbb{P}}[\psi(T)/\psi(t)]$. Further analysis of the pricing kernel properties can be found in Constantinides [16], Hughston [22], Leippold & Wu [32], Shefrin [37], and Rogers [36].
6. The real bond, which is not an asset traded on the market, is defined as the ratio between the inflation bond and the current price index level: $P^R(t, T) = P^I(t, T)/I(t)$.¹³ Both $P^I(t, T)$ and $I(t)$ have been defined previously. Using the same logic seen above, from the real bond prices one can extract a real term structure of interest rates: in particular one can define the real short rate $r(t) = f^R(t, t)$, where $f^R(t, T) = -\partial \log(P^R(t, T))/\partial T$. The process $\{r(t)\}_{t \geq 0}$ is $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted and can be used to define the real money market account $B^R(t)$, with dynamics $dB^R(t) = r(t)B^R(t)dt$: $B^R(t)$ is locally riskless in the real risk-neutral measure \mathbb{Q}^R (as $B(t)$ is the \mathbb{Q} measure). One can also define the real pricing kernel $\psi^R(t) = I(t)\psi(t)$: similarly one can show that $P^R(t, T) = \mathbb{E}_t^{\mathbb{P}}[\psi^R(T)/\psi^R(t)]$. Furthermore, one can introduce the $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted process $\{\lambda^R(t)\}_{t \geq 0}$, called “real market price of risk” and obtain the dynamics: $d\psi^R(t) = -r(t)\psi^R(t)dt - \psi^R(t)\lambda^R(t) \cdot dW^{\mathbb{P}}(t)$.
7. The definition of the real bond, real rates and real pricing kernel is sufficient to define another economy, labelled “real” economy. This is the cornerstone of the so-called “Forex Analogy” (see Hughston [23], Hughston [22], or Brigo & Mercurio [7]): because one can write $I(t) = \psi^R(t)/\psi(t)$, one can see the price index $I(t)$ as the exchange rate between the real and the nominal economy. This allows one to obtain, in analogy with the FX spot rate drift, the risk neutral drift for the price index: $\mathbb{E}_t^{\mathbb{Q}}[I(t+dt) - I(t)] = (n(t) - r(t))I(t)dt$.

¹²Given two n -dimensional vectors a, b , with components a_1, \dots, a_n and b_1, \dots, b_n respectively, the notation $a \cdot b$ is equivalent to: $\sum_{i=1}^n a_i b_i$. This notation is used extensively in this work. Under no circumstances this notation has to be confused with a Stratonovich integral.

¹³It is worth stressing that, although the model proposed in this paper does not require the concept of real bond and real rates, these are often found in the literature, and therefore it is useful to show how these quantities can be recovered in the present set-up.

6.4 Economy dynamics and central bank role

We make some assumptions regarding the economy. These assumptions allow one to make a realistic description of the economy based on a continuous-time model that we showed is based on a widely used discrete-time macroeconomic model, called “Dynamic Stochastic General Equilibrium” model (referred to as DSGE—presented and discussed in the previous section).

1. At time t , the economy is described by three positive quantities $X(t)$, $I(t)$, and $M(t)$, that represent the real output of the economy, the price level in the economy, and the money supply respectively. The real output is an alternative expression for the real Gross Domestic Product, also referred to as real GDP. Money supply is defined as the total amount of cash available in the economy. The processes $\{X(t)\}_{t \geq 0}$, $\{I(t)\}_{t \geq 0}$, and $\{M(t)\}_{t \geq 0}$ are positive $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes.
2. The \mathbb{P} -dynamics of instantaneous output and price index are defined as follows:

$$dX(t) = X(t)[m_X(t)dt + s_X(t) \cdot dW^\mathbb{P}(t)] \quad (33)$$

$$dI(t) = I(t)[m_I(t)dt + s_I(t) \cdot dW^\mathbb{P}(t)] \quad (34)$$

where $m_X(t)$ and $m_I(t)$ are one-dimensional stochastic $\{\mathcal{F}_t\}_{t \geq 0}$ -adapted processes whose dynamics are to be defined below, and $s_X(t)$ and $s_I(t)$ are n -dimensional deterministic processes. These volatilities will be referred to as “model” volatilities in the calibration section, as opposed to volatilities implied by market prices of options. The choice of Brownian motions for the real output and price index relative change processes is reasonable as these quantities are always positive and have historically shown an upward trend with some noise¹⁴. In particular we showed in the previous section that the above two equations for the growth and inflation rate can be derived from a well specified macroeconomic model.

3. The dynamics of the expectations exhibit are modelled using the SDEs:

$$dm_X(t) = a_X(t)dt + b_X(t) \cdot dW^\mathbb{P}(t) \quad (35)$$

$$dm_I(t) = a_I(t)dt + b_I(t) \cdot dW^\mathbb{P}(t) \quad (36)$$

where the processes $a_X(t)$ and $a_I(t)$ are one-dimensional deterministic processes and $b_X(t)$ and $b_I(t)$ are n -dimensional deterministic processes.

4. We assume that the central bank is the only institution responsible for money supply. The central bank uses the money supply as a policy tool and tries to keep the economy close to a target annual growth rate \bar{x} and to a target annual inflation rate \bar{p} . The targets \bar{x} and \bar{p} are constant real numbers. According to standard macroeconomic theory, an increase in money supply can increase both the price level and the output: the central bank can attach more importance to the growth target or to the price stability. The relative importance of these two goals is modelled with the two real positive constants h_x and h_p . To summarise the above assumptions, we assume that the central bank policy is explained by the \mathbb{P} -dynamics

$$dM(t)/M(t) = -h_p(dI(t)/I(t) - \bar{p}dt) - h_x(dX(t)/X(t) - \bar{x}dt) + s_M(t) \cdot dW^\mathbb{P}(t). \quad (37)$$

This curve is also known as “central bank reaction function”.¹⁵ Here $s_M(t)$ is an n -dimensional deterministic process that measures the uncertainty around the central bank policy. These volatilities

¹⁴In order to ensure that this assumption is reasonable, we have run a normality test (Jarque-Bera) to the time series of the Eurozone GDP (quarterly readings) and Consumer Price Index (monthly readings) for the last 30 years. In both cases the normality assumptions is accepted.

¹⁵The above expression for the reaction function attaches more importance to intuition than to formal correctness: if one wanted to write an expression containing only stochastic differentials and not ratios of stochastic differentials (like $dM(t)/M(t)$, $dI(t)/I(t)$, or $dX(t)/X(t)$) one can define the reaction function in logarithmic terms and adjust the equilibrium levels from p and p^* to x and x^* respectively for the change in drifts:

$$d \log M(t) = [-h_p(d \log I(t) - \bar{p}^*dt) - h_x(d \log X(t) - \bar{x}^*dt) + s_M(t) \cdot dW^\mathbb{P}(t)]. \quad (38)$$

will be referred to as “model” volatilities in the calibration section, as opposed to volatilities implied by market prices of options. The above equation can be read as follows: modulo some uncertainty (modelled by the term $s_M(t) \cdot dW(t)$), the central bank will reduce the money supply (both $-h_p$ and $-h_x$ are negative real numbers¹⁶) when inflation or output growth are above their targets. It should be noted that the above specification for the central bank policy is similar to the Taylor rule (see Walsh [42], Woodford [43], Taylor [40], Clarida, Dali & Gertler [14], or Clarida, Dali & Gertler [13]): however, because the Taylor rule assumes that the short term interest rate is the monetary policy tool (as opposed to the money supply), the Taylor rule can lead to negative policy rates, while in a low rates environment central banks tend to use open market operations as policy tools.¹⁷

5. The central bank changes the money supply by trading in the secondary bond market, which has some feedback effects on bond prices. These effects are known by market participants. The central bank can also target some specific sectors of the yield curve, for example it can decide to sell short maturity bonds and buy longer maturities bonds to make the curve flatter while not inflating its balance sheet.¹⁸ We assume that the relative increase in the money supply has a linear relationship with the relative increase in the bond prices, weighted for each maturity T by a weight function $Z(T)$. These effects are priced by the market and are modelled by the \mathbb{Q} -dynamics for the money supply:

$$dM(t)/M(t) = \gamma dt + \int_t^{t+\Omega} Z(T)[dP(t, T)/P(t, T)]dT + s_L(t) \cdot dW^{\mathbb{Q}}(t). \quad (39)$$

Here γ is a real constant that models the natural growth of the money supply; $Z(T)$ is a real, positive and increasing deterministic function of the bond maturity T . The request that the function $Z(T)$ is always positive is a request from economic assumptions: if bond prices increase, nominal rates decrease, which is equivalent to saying that the money supply goes up; in this framework the interest rate itself is not the policy tool, because all monetary policy is modelled via the money supply $M(t)$. The integral in the above expression is a deterministic one, given that at time t the quantity $dP(t, T)/P(t, T)$ is known and therefore deterministic: the integral in the above expression has to be regarded as a way to weight the impact of relative changes in the bond prices across the different maturities $T \in [t, t + \Omega]$ of the term structure.¹⁹ The real positive constant $\Omega > 0$ represents the time horizon used by the central bank to purchase or sell nominal bonds in order to influence the money supply $M(t)$. For example, if the central bank is trading bonds up to the 30 years maturity, one sets the parameter Ω to 30.

Uncertainty around this relationship is captured by the stochastic differential, multiplied by a liquidity volatility deterministic n -dimensional process $\{s_L(t)\}_{t \geq 0}$.

6. We finally require the following relationship to hold:

$$h_p s_I(t) + h_x s_X(t) - s_M(t) = 0 \quad (41)$$

This condition is equivalent to asking that the central bank reaction function and the liquidity relationship are locally riskless: this condition is required to ensure that the central bank policy is credible,

In the rest of the paper we will not be using the above expression and will develop our theory using 37.

¹⁶Because the so-called “quantitative easing” has been implemented only in the last few years by some central banks, it is not possible to provide data-based estimates of these parameters for the moment.

¹⁷This consideration is even more relevant in the current low rates environment, where the main option left to the central banks in the USA, UK, Japan and the Euro area is to purchase bonds to stimulate and reflate the economy (so-called “quantitative easing”).

¹⁸The recent “operation twist” implemented by the FED is a good example.

¹⁹An observation similar to the comment made on the reaction function can be done at this stage: the liquidity relationship can be rewritten as:

$$d \log M(t) = [\gamma dt + \int_t^{t+\Omega} Z(T)[d \log P(t, T) + 1/2[\sigma_P(t, T) \cdot P(t, T)]^2]dT + s_L(t) \cdot dW^{\mathbb{Q}}(t)]. \quad (40)$$

We will not be using the above expression in the model theory development, but rather 39.

because the monetary policy is adjusted immediately following the random shocks to the economy. It should be noted that the reaction function is still stochastic as the drifts are stochastic, and that the liquidity relationship is stochastic as the short rate is stochastic. We also note that the above condition ensures that the diffusion term for the \mathbb{P} -dynamics 37 is the same as the diffusion term for the \mathbb{Q} -dynamics 39, therefore satisfying Girsanov's theorem.

7 CTCB Model construction

We now build the pricing model, which will be referred to as “Continuous-time central bank” (CTCB) model in the following sections. The assumptions made so far can be regarded as standard no-arbitrage assumptions in the financial market, in conjunction with some reasonable assumptions on growth and inflation rate (modelled as Brownian motions with some stochastic drifts — historically GDP and price levels have shown an upward trend with some noise). Furthermore, the central bank trades nominal bonds to keep the economy around some target levels, and this has some (wanted) effects on the bond prices, and hence on the yield curve.

The model construction that follows puts together the financial market and macroeconomic assumptions to obtain a pricing framework that is consistent both with the economic theory and the no-arbitrage principle.

Step 1 - The \mathbb{Q} -dynamics of the economic variables and expectations are obtained with Girsanov theorem:

$$dm_X(t) = (a_X(t) - \lambda(t) \cdot b_X(t))dt + b_X(t) \cdot dW^{\mathbb{Q}}(t) \quad (42)$$

$$dm_I(t) = (a_I(t) - \lambda(t) \cdot b_I(t))dt + b_I(t) \cdot dW^{\mathbb{Q}}(t) \quad (43)$$

$$dX(t)/X(t) = (m_X(t) - \lambda(t) \cdot s_X(t))dt + s_X(t) \cdot dW^{\mathbb{Q}}(t) \quad (44)$$

$$dI(t)/I(t) = (m_I(t) - \lambda(t) \cdot s_I(t))dt + s_I(t) \cdot dW^{\mathbb{Q}}(t). \quad (45)$$

Step 2 - Similarly, the \mathbb{Q} -dynamics for the central bank policy are obtained using Girsanov theorem:

$$dM(t)/M(t) = -h_p(dI(t)/I(t) - \bar{p}dt) - h_x(dX(t)/X(t) - \bar{x}dt) - \lambda(t) \cdot s_M(t)dt + s_M(t) \cdot dW^{\mathbb{Q}}(t). \quad (46)$$

Step 3 - Putting together the central bank policy equation 46 and the economy dynamics (equations 44 and 45) in the risk neutral measure we get:

$$\begin{aligned} dM(t)/M(t) = & -h_p((m_I(t) - \lambda(t) \cdot s_I(t))dt + s_I(t) \cdot dW^{\mathbb{Q}}(t) - \bar{p}dt) \\ & - h_x((m_X(t) - \lambda(t) \cdot s_X(t))dt + s_X(t) \cdot dW^{\mathbb{Q}}(t) - \bar{x}dt) - \lambda(t) \cdot s_M(t)dt + s_M(t) \cdot dW^{\mathbb{Q}}(t). \end{aligned} \quad (47)$$

Step 4 - Equating the central bank policy equation 47 and equation 39, which models the impact of central bank policy on bond prices, we obtain:

$$\begin{aligned} \gamma dt + \int_t^{t+\Omega} Z(T)[dP(t, T)/P(t, T)]dT + s_L(t) \cdot dW^{\mathbb{Q}}(t) = & -h_p((m_I(t) - \lambda(t) \cdot s_I(t))dt + s_I(t) \cdot dW^{\mathbb{Q}}(t) - \bar{p}dt) \\ & - h_x((m_X(t) - \lambda(t) \cdot s_X(t))dt + s_X(t) \cdot dW^{\mathbb{Q}}(t) - \bar{x}dt) - \lambda(t) \cdot s_M(t)dt + s_M(t) \cdot dW^{\mathbb{Q}}(t). \end{aligned} \quad (48)$$

Step 5 - Combining the above and the no arbitrage condition for the bond price dynamics (32), we obtain:

$$\begin{aligned} \gamma dt + \int_t^{t+\Omega} Z(T)[n(t)dt + \sigma_P(t, T) \cdot dW^{\mathbb{Q}}(t)]dT + s_L(t) \cdot dW^{\mathbb{Q}}(t) = & -h_p((m_I(t) - \lambda(t) \cdot s_I(t))dt + s_I(t) \cdot dW^{\mathbb{Q}}(t) - \bar{p}dt) \\ & - h_x((m_X(t) - \lambda(t) \cdot s_X(t))dt + s_X(t) \cdot dW^{\mathbb{Q}}(t) - \bar{x}dt) - \lambda(t) \cdot s_M(t)dt + s_M(t) \cdot dW^{\mathbb{Q}}(t). \end{aligned} \quad (49)$$

We compact the notation by introducing the following quantities:

$$\zeta(t) = \int_t^{t+\Omega} Z(T) dT$$

$$\Sigma_P(t) = \int_t^{t+\Omega} Z(T) \sigma_P(t, T) dT$$

We note that the quantity $\zeta(t)$ is always strictly positive because the function $Z(T)$ and the constant Ω are requested to be strictly positive. With the above definitions the model equation becomes:

$$\begin{aligned} \zeta(t)n(t)dt + \Sigma_P(t) \cdot dW^{\mathbb{Q}}(t) = & -h_p[(m_I(t) - \lambda(t) \cdot s_I(t))dt + s_I(t) \cdot dW^{\mathbb{Q}}(t) - \bar{p}dt] \\ & -h_x[(m_X(t) - \lambda(t) \cdot s_X(t))dt + s_X(t) \cdot dW^{\mathbb{Q}}(t) - \bar{x}dt] - \lambda(t) \cdot s_M(t)dt + s_M(t) \cdot dW^{\mathbb{Q}}(t) - \gamma dt - s_L(t) \cdot dW^{\mathbb{Q}}(t) \end{aligned}$$

Step 6 - The no-arbitrage conditions are obtained from the above equation by collecting the terms multiplied by dt and $dW(t)$ in the following way:

$$\gamma = h_p \bar{p} + h_x \bar{x} \quad (50)$$

$$\zeta(t)n(t) = -h_p[m_I(t) - \lambda(t) \cdot s_I(t)] - h_x[m_X(t) - \lambda(t) \cdot s_X(t)] - \lambda(t) \cdot s_M(t) \quad (51)$$

$$\Sigma_P(t) = \int_t^{t+\Omega} z(T) \sigma_P(t, T) dT = -h_p s_I(t) - h_x s_X(t) + s_M(t) - s_L(t). \quad (52)$$

We note that we can equate the equation above to $s_L(t)$ thanks to condition 39:

$$\Sigma_P(t) = \int_t^{t+\Omega} z(T) \sigma_P(t, T) dT = -h_p s_I(t) - h_x s_X(t) + s_M(t) - s_L(t) = -s_L(t). \quad (53)$$

Some observations can be made regarding these non-arbitrage conditions:

1. The first condition can be seen as the risk-neutral drift for the money supply assuming that there is no uncertainty and no monetary policy. Here we group all deterministic terms multiplied by dt . Therefore we can refer to the constant γ as the natural money supply growth rate. We also note that the constant γ is likely to be positive, given that the central bank reaction function parameters h_x and h_p are positive by construction and that the target levels \bar{x} and \bar{p} are normally positive numbers. This matches the intuition that over time the money supply tends to grow, unless the central bank tries to reduce it.
2. The second calibration conditions gives us a closed-form expression for the short rate that will be used in the following section to get to some short rate dynamics. It remembers the condition 41

$$h_p s_I(t) + h_x s_X(t) - s_M(t) = 0$$

the calibration condition simplifies into

$$\zeta(t)n(t) = -h_p m_I(t) - h_x m_X(t) \quad (54)$$

which shows that the second calibration condition contains all stochastic terms multiplied by dt .

3. The bond volatilities $\sigma_P(t, T)$ are determined by other model parameters and can be regarded as a combination of the economic factor volatilities. This is clear from the third calibration condition, which contains all terms multiplied by $dW(t)$. There are constraints on $s_M(t)$ (see condition 41) that allow one to write $\Sigma_P(t) = -s_L(t)$.

Step 7 - To price inflation derivatives, it can be useful to work with the T^* -forward measure. By using the techniques detailed for example in Brigo & Mercurio [7], one obtains the inflation and bond price dynamics under this measure:

$$dI(t)/I(t) = (m_I(t) - \lambda(t) \cdot s_I(t) + \sigma_P(t, T^*) \cdot s_I(t))dt + s_I(t) \cdot dW^{T^*}(t). \quad (55)$$

$$dP(t, T)/P(t, T) = (n(t) + s_P(t, T) \cdot \sigma_P(t, T^*))dt + s_P(t, T) \cdot dW^{T^*}(t). \quad (56)$$

Having an explicit form for the pricing kernel allows one to price derivatives using the real world measure, if needed.

We close this section with an observation on instantaneous correlations. In this model the same n -dimensional Brownian motion is the source of randomness for all variables. If one wanted to see what are the instantaneous correlations implied by this choice one remembers that, given two stochastic differential equations $dX(t) = a dW^1(t) + b dW^2(t)$ and $dY(t) = c dW^1(t) + f dW^2(t)$ (where $\{W^1(t)\}_{t \geq 0}$ and $\{W^2(t)\}_{t \geq 0}$ are two independent one-dimensional Brownian motions and a, b, c , and f are deterministic real constants), one obtains $dX(t)dY(t) = (ac + bf)dt$. Perhaps we can write a general formula for the instantaneous correlation ρ_t using quadratic variations and covariation $\rho_t = \frac{d\langle X, Y \rangle_t}{\sqrt{d\langle X, X \rangle_t} \sqrt{d\langle Y, Y \rangle_t}} = \frac{(ac + bf)}{\sqrt{a^2 + b^2} \sqrt{c^2 + f^2}}$. Therefore one can in principle use the model volatilities to calibrate also market-implied instantaneous correlations between the macroeconomic variables.

8 Equivalent interest rates model

Here we show that the model presented in the previous section, although is a completely new model and is derived from macroeconomic assumptions, yields a model for the short rate that can be seen as a mean-reverting Hull-White model. The Hull-White model, presented in Hull & White [26] and further analysed in Brigo & Mercurio [7], is a widely-used model for the short rate $n(t)$ that has the properties of being mean-reverting and calibrating to any given term structure of interest rates. Here we show how this model is derived within the macroeconomic framework and study its mean-reverting property as a function of the economy parameters.

The derivation is carried out as follows. The second calibration condition 51 gives an expression containing the short term interest rate $n(t)$:

$$\zeta(t)n(t) = -h_p[m_I(t) - \lambda(t) \cdot s_I(t)] - h_x[m_X(t) - \lambda(t) \cdot s_X(t)] - \lambda(t) \cdot s_M(t)$$

If one differentiates this condition and remembers the condition 41 one gets:

$$d\zeta(t)n(t) + \zeta(t)dn(t) = -h_p dm_I(t) - h_x dm_X(t)$$

There is no covariance term in the above differential given that $\zeta(t)$ is a deterministic quantity. One can remember the expressions for the drift differentials 42 and 43 and substitute them in the above expression, obtaining:

$$d\zeta(t)n(t) + \zeta(t)dn(t) = -h_p[[a_I(t) - \lambda(t) \cdot b_I(t)]dt + b_I(t) \cdot dW^Q(t)] - h_x[[a_X(t) - \lambda(t) \cdot b_X(t)]dt + b_X(t) \cdot dW^Q(t)]$$

Further, one needs to calculate the differential of $\zeta(t)$:

$$d\zeta(t) = \left(\frac{\partial \int_t^{t+\Omega} Z(T) dT}{\partial t} \right) dt = [Z(t + \Omega) - Z(t)]dt.$$

One substitutes the above in the differential, after rearranging one obtains:

$$\zeta(t)dn(t) = -[Z(t + \Omega) - Z(t)]n(t)dt - h_p[[a_I(t) - \lambda(t) \cdot b_I(t)]dt + b_I(t) \cdot dW^Q(t)] +$$

$$-h_x[[a_X(t) - \lambda(t) \cdot b_X(t)]dt + b_X(t) \cdot dW^{\mathbb{Q}}(t)]$$

We can now express the differential of the short rate $n(t)$:

$$\begin{aligned} dn(t) = & -[Z(t + \Omega) - Z(t)]/\zeta(t)n(t)dt - h_p/\zeta(t)[[a_I(t) - \lambda(t) \cdot b_I(t)]dt \\ & + b_I(t) \cdot dW^{\mathbb{Q}}(t)] - h_x/\zeta(t)[[a_X(t) - \lambda(t) \cdot b_X(t)]dt + b_X(t) \cdot dW^{\mathbb{Q}}(t)]. \end{aligned}$$

If one defines the following terms:

$$f_2(t) = [Z(t + \Omega) - Z(t)]/\zeta(t) \quad (57)$$

$$f_1(t) = [-h_p a_I(t) - h_x a_X(t)]/\zeta(t) \quad (58)$$

$$\sigma_n(t) = [-h_x b_X(t) - h_p b_I(t)]/\zeta(t) \quad (59)$$

This shows that the model implies some short nominal interest rates dynamics that are similar to the ones assumed by the generalised Vasicek model:

$$dn(t) = [f_1(t) - f_2(t)n(t) - \lambda(t) \cdot \sigma_n(t)]dt + \sigma_n(t) \cdot dW^{\mathbb{Q}}(t)$$

It is important to notice that the requests made on the function $Z(T)$ (to be an increasing and positive function) imply that $f_2(t)$ is always positive, i.e. that the nominal short rate is mean-reverting. Before doing some further analysis, we notice that the source of randomness in the CTCB model is n -dimensional, and the volatility function $\sigma_n(t)$ is n -dimensional accordingly. To mark the difference against the original Hull-White model, where the driving Brownian motion is scalar, we write the scalar Hull-White volatility as $\sigma_n^*(t)$: one can link the two processes by asking that the total variance of the source of randomness of the CTCB model is the same as the total variance of the Hull-White model. The relationship is:

$$[\sigma_n^*(t)]^2 = \sum_{i=1}^n [\sigma_n^i(t)]^2 \quad (60)$$

where $\sigma_n^i(t)$ are the single components of the volatility function $\sigma_n(t)$.

Consistently with the Hull-White model, the distribution of the short rate is Gaussian and can generate negative short nominal rates: in the current low rates environment, when central banks are explicitly talking about negative deposit rates, we don't think this is a theoretical problem, rather we think that this model is probably better suited than other positive rates models to deal with the current market conditions. In Denmark the central bank set the short interest rate to -0.2%: in practice central banks can set a negative short rate to stimulate commercial banks to lend to consumers and firms in times of economic distress.

The only differences w.r.t. to the original extended Vasicek model is that the volatility is a multidimensional function of time, and that the driving source of randomness is a multidimensional Brownian motion: as explained in detail in point 4 below, one can find the volatility vector components to target a certain level of total volatility, and therefore the marginal distribution. However these differences do not prevent us from reaching the following conclusions:

1. If we only want to use this model to price interest rates derivatives, one calibrates the function $f_1(t)$ to the nominal forward rates observed in the market, as suggested in the original Hull-White paper, modulo some changes. Alternatively, one can use the calibration condition $\mathbb{E}_t^{T*}[n(T)] = f(t, T)$.
2. The nominal bond price is such that the volatility of the relative moves is a deterministic volatility function. This is no surprise given the original assumptions. This is important because it can simplify the calculation of the year-on-year convexity adjustment, as it will be shown in the following sections.
3. Thanks to the above fact one can use Black-Scholes formulas to price European bond options. Moving to the relevant forward measure allows one to carry out discounting by simply multiplying by the market bond prices. Because it is trivial to price bond options in this model, one can write Black-type

formulas for bond options. These yield closed forms for nominal cap/floors and swaptions (using the method presented in Jamshidian [28]): this intuition will be developed in the following sections.

4. Because the process for the short rate is normal, trees can be easily constructed. In fact the n -dimensional Brownian motion can be treated as a one-dimensional process for this purpose (this technique is also called "flattening", where the independent components of the Brownian motion are summed and considered as a single Brownian motion with the appropriate diffusion term).

9 Further analysis on the mean-reversion property

9.1 General case

Here we analyse the mean-reversion coefficient found in the previous section (result 57) and we link it to the general theory of mean-reverting Gaussian models developed by Hull & White [26]. Here we refer to the original formulation of the model, where the driving Brownian motion $\sigma_n^*(t)$ is a scalar process: one can translate it into vector by using 60 This is not a major problem, as one can "flatten" the vector volatility into an equivalent scalar volatility that leaves the total variance unchanged.

In particular, in that paper the authors present a version of the model with time-dependent coefficients, where the dynamics of the short rate are governed by the SDE:

$$dn(t) = [\theta(t) - a(t)n(t) - \lambda(t)\sigma_n(t)]dt + \sigma_n^*(t)dW^{\mathbb{Q}}(t) \quad (61)$$

They suggest a calibration strategy that yields the model parameters as functions of two functions used to fit the term structure of interest rates using the ansatz $P(t, T) = A(t, T)e^{-n(t)B(t, T)}$. At the initial time $t = 0$ the positive functions $A(0, T)$ and $B(0, T)$ are numerically calibrated to the market term structure $P(0, T)$. In particular, the Hull & White find that the mean reversion speed $a(t)$ has to satisfy the condition:

$$a(t) = -\frac{\frac{\partial B^2(0, t)}{\partial t^2}}{\frac{\partial B(0, t)}{\partial t}} = -\frac{\beta''(t)}{\beta'(t)} \quad (62)$$

where we have made the notation lighter by defining: $\beta(t) = B(0, t)$.

Further, the authors prove a calibration condition for the mean reversion level parameter $\theta(t)$:

$$\theta(t) = \lambda(t)\sigma_n^*(t) - a(t)\frac{\partial \log A(0, t)}{\partial t} - \frac{\partial^2 \log A(0, t)}{\partial t^2} + \left[\frac{\partial B(0, t)}{\partial t}\right]^2 \int_0^t \left[\frac{\sigma_n^*(s)}{\frac{\partial B(0, s)}{\partial s}}\right]^2 ds. \quad (63)$$

At this stage we observe that the time-dependent version of the Hull-White model does not necessarily imply mean reversion: in fact, the mean reversion speed coefficient $a(t)$ is positive (i.e. there is mean reversion) only if the sign of the first derivative $\beta'(t)$ is different from the sign of the second derivative $\beta''(t)$.

For example, if one takes $A(0, t) = 1$, which is a legitimate choice, the function $B(0, t)$ will be increasing where the term structure is upward sloped. Let us introduce the compound spot rate for maturity T observed at time t and denote it by $Y(t, T)$: it is defined as the flat interest rate such that: $P(t, T) = e^{-Y(t, T)(T-t)}$. If $A(0, t) = 1$, this expression has to be equal to $P(0, T) = e^{-n(0)B(0, T)}$. Equalling the two terms one gets $Y(0, T) = n(0)\frac{B(0, T)}{T}$.

We focus our attention on the mean-reversion speed, and equate the result from Hull & White [26] to the expression found in the previous section. This allows one to draw some conclusions on the function $Z(T)$. By equalling 57 and 62 one gets:

$$-\frac{\beta''(t)}{\beta'(t)} = [Z(t + \Omega) - Z(t)]/\zeta(t)$$

$$\int_t^{t+\Omega} Z(T)dT = -[Z(t+\Omega) - Z(t)] \frac{\beta'(t)}{\beta''(t)}.$$

We take a derivative of the above expression w.r.t. t :

$$[Z(t+\Omega) - Z(t)] = -[Z'(t+\Omega) - Z'(t)] \frac{\beta'(t)}{\beta''(t)} - [Z(t+\Omega) - Z(t)] \frac{(\beta''(t))^2 - \beta'(t)\beta'''(t)}{(\beta''(t))^2}.$$

One can rearrange the above expression as:

$$\frac{[Z(t+\Omega) - Z(t)]}{[Z'(t+\Omega) - Z'(t)]} = -\frac{\frac{\beta'(t)}{\beta''(t)}}{\left(\frac{2(\beta''(t))^2 - \beta'(t)\beta'''(t)}{(\beta''(t))^2}\right)} = -\frac{\beta'(t)\beta''(t)}{2(\beta''(t))^2 - \beta'(t)\beta'''(t)}.$$

By defining $u(t) = Z(t+\Omega) - Z(t)$ and $\alpha(t) = -\frac{\beta'(t)\beta''(t)}{2(\beta''(t))^2 - \beta'(t)\beta'''(t)}$ and requiring that $\beta'(t) \neq 0$, $\beta''(t) \neq 0$, one rewrites the above expression as:

$$\frac{u'(t)}{u(t)} = \frac{d \log u(t)}{dt} = \alpha^{-1}(t).$$

One recalls that the first two conditions $\beta'(t) \neq 0$, $\beta''(t) \neq 0$ have already been seen in the above examples and are equivalent to requiring that the term structure is not flat (a trivial case that never happens in practice) and that the term structure is not explosive, so that mean reversion can be ensured.

This linear ODE is solved in $t > t_0$ to yield $u(t) = u(t_0)e^{\int_{t_0}^t \alpha^{-1}(s)ds}$.

The meaning of the above result is a relationship between the functions $Z(t)$ and $\beta(t)$:

$$[Z(t+\Omega) - Z(t)] = [Z(t_0+\Omega) - Z(t_0)]e^{\int_{t_0}^t -\frac{2(\beta''(s))^2 - \beta'(s)\beta'''(s)}{\beta'(s)\beta''(s)}ds}. \quad (64)$$

One can make two observations from this expression:

1. To ensure mean reversion, one has to require that the function $Z(t)$ is positive and increasing: this request was made in the previous section and is now confirmed by looking at the properties of the Hull-White model, in particular 64. In fact, if $Z(t)$ is positive and increasing, $Z(t_0+\Omega) > Z(t_0)$, an exponential is always positive, being consistent with the request $Z(t+\Omega) > Z(t)$.
2. One can use the relationship above to calibrate the function $Z(t)$ as a function of $\beta(t)$ and $A(0, t)$ (i.e. the term structure of interest rates), or one can set the functions $A(0, t)$ and $Z(t)$ using some functional forms and obtain the function $\beta(t)$ by calibrating to the term structure. The latter seems more appropriate, as we may want to make some assumptions on the market liquidity function $Z(t)$.

The analysis done so far allows one also to impose a further calibration constraint on the model bond volatilities. In the previous section we have introduced the bond volatilities $\sigma_P(t, T)$ without specifying more details: now, taking in consideration result 59 and the bond volatility formula in the Hull-White model, one can impose a further calibration condition:

$$\sigma_P(t, T) = [-h_x b_X(t) - h_P b_I(t)]/\zeta(t) \left[\frac{\beta(T) - \beta(t)}{\beta'(t)} \right]. \quad (65)$$

9.2 Constant mean reversion speed

We conclude this section with an observation regarding the constant mean reversion speed of the Hull White model, that is used in many applications. The main result we find is that if one imposes that the function

$Z(t)$ is an exponential in the form $Z(t) = e^{\delta t}$ (with $\delta > 0$ to ensure that $Z(t)$ is increasing), one immediately shows that the mean reversion speed has the property:

$$[Z(t + \Omega) - Z(t)]/\zeta(t) = \frac{Z(t + \Omega) - Z(t)}{\int_t^{t+\Omega} Z(T) dT} = \delta. \quad (66)$$

This result is interesting from a theoretical perspective, because a higher mean reversion speed decreases the intra-curve rates correlation, as it is well known in literature, and this has a similar meaning as increasing the parameter δ : a higher parameter δ means that the longer maturities of the curve react more strongly to monetary policy compared to the short end of the curve, therefore increasing the intra-curve decorrelation. However this result has a very practical implication too: the relationship 64 can be difficult to implement numerically, as one wants to impose the liquidity function $Z(t)$ and imply $\beta(t)$ (the converse would be trivial): this could reduce the flexibility of the model.

However, we remember result 62 and obtain the differential equation $\delta = -\frac{\beta''(t)}{\beta'(t)}$, which is solved by $\beta(t) = e^{-\delta t}$. Therefore, known $Z(t)$, one has $B(0, t) = \beta(t)$. By observing from the market the short rate $n(0)$ and the term structure $P(0, t)$, one uses the ansatz $P(0, T) = A(0, T)e^{-n(0)B(0, T)}$ to obtain $A(0, t) = \frac{P(0, t)}{e^{-n(0)e^{-\delta t}}}$, which fully calibrates the model as far as nominal rates are concerned. This result will be exploited in the calibration process in the following section. From this point we will assume that $Z(T) = e^{\delta T}$. By doing some calculations one shows that the form $Z(t) = e^{\delta T}$ satisfies the relationship 64:

$$[Z(t + \Omega) - Z(t)] = [Z(t_0 + \Omega) - Z(t_0)]e^{\int_{t_0}^t -\frac{2(\beta''(s))^2 - \beta'(s)\beta'''(s)}{\beta'(s)\beta''(s)} ds}$$

$$e^{\delta(t+\Omega)} - e^{\delta t} = [e^{\delta(t_0+\Omega)} - e^{\delta t_0}]e^{\int_{t_0}^t \frac{2\delta^4 - \delta^4}{\delta^3} ds} = [e^{\delta(t_0+\Omega)} - e^{\delta t_0}](e^{\delta(t-t_0)}).$$

Finally it is useful to show how the choice of $Z(t) = e^{\delta T}$ is consistent with the classical integration of the Ornstein-Uhlenbeck process: in fact, given the SDE $dn(t) = [\theta(t) - an(t)]dt + \sigma_n^*(t)dW(t)$ with known initial condition $n(s)$, one writes the differential for $n(t)e^{at}$, integrates and obtain the standard result:

$$n(t) = n(s)e^{-a(t-s)} + \int_s^t e^{-a(t-u)}\theta(u)du + \int_s^t e^{-a(t-u)}\sigma_n^*(u)dW(u).$$

In the above formula $\sigma_n^*(u)$ refers to the Hull-White scalar short rate volatility. If one now remembers the calibration condition 51 one can try and substitute the expression for $\zeta(t)$ inside it and confirm that one gets the same result stated in the above formula. In fact:

$$\zeta(t) = \int_t^{t+\Omega} Z(T) dT = \int_t^{t+\Omega} e^{\delta T} dT = \frac{e^{\delta(t+\Omega)} - e^{\delta t}}{\delta} = \frac{e^{\delta t}(e^{\delta\Omega} - 1)}{\delta}.$$

If one recalls the calibration condition 51 and the condition 41 one writes:

$$\begin{aligned} \zeta(t)n(t) &= -h_p[m_I(t) - \lambda(t) \cdot s_I(t)] - h_x[m_X(t) - \lambda(t) \cdot s_X(t)] - \lambda(t) \cdot s_M(t) = -h_p m_I(t) - h_x m_X(t) \\ \frac{e^{\delta t}(e^{\delta\Omega} - 1)}{\delta} n(t) &= -[h_p m_I(s) + h_x m_X(s)] - \int_s^t [h_p a_I(u) + h_x a_X(u)] du - \int_s^t [h_p b_I(u) + h_x b_X(u)] \cdot dW(u) \\ n(t) &= n(s)e^{-\delta(t-s)} - \int_s^t \frac{[h_p a_I(u) + h_x a_X(u)]\delta e^{-\delta(t-u)}}{(e^{\delta\Omega} - 1)} \frac{1}{e^{\delta u}} du - \int_s^t \frac{[h_p b_I(u) + h_x b_X(u)]\delta e^{-\delta(t-u)}}{(e^{\delta\Omega} - 1)} \frac{1}{e^{\delta u}} \cdot dW(u) \end{aligned}$$

where we recall the calibration condition $n(s) = \frac{-[h_p m_I(s) + h_x m_X(s)]}{\zeta(s)}$. Further calculations yield:

$$n(t) = n(s)e^{-\delta(t-s)} - \int_s^t \frac{[h_p a_I(u) + h_x a_X(u)]e^{-\delta(t-u)}}{\zeta(u)} du - \int_s^t \frac{[h_p b_I(u) + h_x b_X(u)]\delta e^{-\delta(t-u)}}{\zeta(u)} \cdot dW(u)$$

If one remembers the definitions 58 and 59, one finally obtain the desired result:

$$n(t) = n(s)e^{-\delta(t-s)} + \int_s^t e^{-\delta(t-u)} \theta(u) du + \int_s^t e^{-\delta(t-u)} \sigma_n(u) \cdot dW(u).$$

In the above formula we stress that now the function $\sigma_n(u)$ is the CTCB vector short rate volatility.

10 Pricing of vanilla interest rates derivatives

Finding that our macro-based CTCB model yields a short rate model that is a Hull-White model makes the pricing of interest rates derivatives much simpler. The main result is that, because bond prices are lognormally distributed, one can use Black-type formulae to price bond options. Bond options are used also to find the prices of caplets and floorlets, therefore allowing one to price caps and floors: this is explained for example in Brigo & Mercurio [7]. For swaptions, the method suggested by Jamishidian [28] can be followed. Closed forms allow faster pricing of vanilla interest rates derivatives, therefore speeding up the calibration. We quote some results that are useful and that can be found for example in Hull & White [26] and Brigo & Mercurio [7]. In the following calculations we use the quantity $P(t, T_1, T_2)$, defined as a portfolio of a long bond $P(t, T_2)$ and a short bond $P(t, T_1)$.

Lemma 1 *The undiscounted price of a vanilla option on a lognormally distributed asset $X(T)$ with strike K , whose logarithm has expectation $\mathbb{E}[\log X(T)] = M$ and variance $\text{Var}[\log X(T)] = V^2$, is given by:*

$$\mathbb{E}[\omega(X - K)^+] = \omega e^{M+1/2V^2} N(\omega(M - \log(K) + V^2)/V) - \omega K N(\omega(M - \log(K))/V) \quad (67)$$

where the function $N(x)$ is the cumulative standard Gaussian distribution, i.e. $N(x) = \int_{-\infty}^x (2\pi)^{-1} e^{-\frac{t^2}{2}} dt$ and $\omega \in \{-1, 1\}$ for puts and calls respectively.

Lemma 2 *The variance between times t and T_1 of the quantity $P(t, T_1, T_2)$, with $t < T_1 < T_2$, in the Hull-White model is*

$$V_P(t, T_1, T_2) = [\beta(T_2) - \beta(T_1)]^2 \int_t^{T_1} \left[\frac{\sigma_n^*(u)}{\beta'(u)} \right]^2 du. \quad (68)$$

Lemma 3 *The price at time t of an option on the quantity $P(t, T_1, T_2)$ with option maturity T_1 and option strike K in the Hull-White model with $t < T_1 < T_2$ is*

$$ZBO(\text{call}, t, T_1, T_2, K) = P(t, T_2) N(h) - K P(t, T_1) N(h - (V_P(t, T_1, T_2))^{\frac{1}{2}}) \quad (69)$$

$$ZBO(\text{put}, t, T_1, T_2, K) = -P(t, T_2) N(-h) + K P(t, T_1) N((V_P(t, T_1, T_2))^{\frac{1}{2}} - h) \quad (70)$$

where $h = \frac{1}{V_P(t, T_1, T_2)^{\frac{1}{2}}} \log \left[\frac{P(t, T_2)}{P(t, T_1)K} \right] + \frac{V_P(t, T_1, T_2)^{\frac{1}{2}}}{2}$.

Lemma 4 *The undiscounted price at time t of a caplet (floorlet) with maturity T_1 , strike K , and notional M , on the forward rate between times T_1 and T_2 , denoted as $F(t, T_1, T_2)$, is the price of a put (call) option with strike $(1 + K(T_2 - T_1))^{-1}$, notional $M(1 + K(T_2 - T_1))$, maturity T_1 on the quantity $P(t, T_1, T_2)$ between times T_1 and T_2 . The result is model independent and we assume $t < T_1 < T_2$.*

$$\text{Caplet}(t, T_1, T_2, K, N) = M(1 + K(T_2 - T_1)) ZBO(\text{put}, t, T_1, T_2, (1 + K(T_2 - T_1))^{-1}) \quad (71)$$

$$\text{Floorlet}(t, T_1, T_2, K, N) = M(1 + K(T_2 - T_1))ZBO(\text{call}, t, T_1, T_2, (1 + K(T_2 - T_1))^{-1}) \quad (72)$$

Lemma 5 *The price of a coupon bearing bond option in the Hull-White model is equivalent to pricing a portfolio of zero-coupon bond options prices using the special nominal rate n^* .*

$$CBO(\text{call}, t, T, T_1 \dots T_M, c_1 \dots c_M, K) = \sum_{i=1}^M c_i ZBO(\text{call}, t, T_{i-1}, T_i, P(t, T_i, n^*)) \quad (73)$$

$$CBO(\text{put}, t, T, T_1 \dots T_M, c_1 \dots c_M, K) = \sum_{i=1}^M c_i ZBO(\text{put}, t, T_{i-1}, T_i, P(t, T_i, n^*)) \quad (74)$$

Lemma 6 *The price at time t of a payer (P) swaption (that gives the right to enter into a payer swaption with fixed rate K and payment dates T_i) is equivalent to the price of a coupon bearing bond option. The result is model-independent.*

$$\text{Swtpn}(P, t, T, T_1 \dots T_M, K) = CBO(\text{put}, t, T, T_1 \dots T_M, c_1 \dots c_M, K) = \sum_{i=1}^M c_i ZBO(\text{put}, t, T_{i-1}, T_i, X_i) \quad (75)$$

$$\text{Swtpn}(R, t, T, T_1 \dots T_M, K) = CBO(\text{call}, t, T, T_1 \dots T_M, c_1 \dots c_M, K) = \sum_{i=1}^M c_i ZBO(\text{call}, t, T_{i-1}, T_i, X_i) \quad (76)$$

$$X_i = A(T, T_i)e^{-n^* B(T, T_i)}$$

Lemma 7 *The price of a swaption with strike K , maturity T and payment dates $T_i > T$ in the Hull-White model is:*

$$S(P, t, T, T_1 \dots T_M, K) = \sum_{i=1}^M c_i ZBO(\text{put}, t, T_{i-1}, T_i, X_i) = \sum_{i=1}^M c_i [-P(t, T_i)N(-h_i) + X_i P(t, T_{i-1})N((V_P(t, T_{i-1}, T_i))^{\frac{1}{2}} - h_i)] \quad (77)$$

$$S(R, t, T, T_1 \dots T_M, K) = \sum_{i=1}^M c_i ZBO(\text{call}, t, T_{i-1}, T_i, X_i) = \sum_{i=1}^M c_i [P(t, T_i)N(h_i) - X_i P(t, T_{i-1})N(h_i - (V_P(t, T_{i-1}, T_i))^{\frac{1}{2}})] \quad (78)$$

$$h_i = \frac{1}{V_P(t, T_i, T_{i-1})^{\frac{1}{2}}} \log \left[\frac{P(t, T_i)}{P(t, T_{i-1})X_i} \right] + \frac{V_P(t, T_{i-1}, T_i)^{\frac{1}{2}}}{2}$$

$$X_i = A(T, T_i)e^{-n^* B(T, T_i)}.$$

We can now start pricing derivatives based on the above results using the macroeconomic model defined in the previous sections and leveraging on the equivalent short rate model.

Lemma 8 *The price of bond options, caplets and floorlets, and swaptions in the CTCB model follow the formulas seen above with the following parametrisation*

$$[\sigma_n^*(t)]^2 = \sum_{i=1}^N \{[-h_x b_X^i(t) - h_p b_I^i(t)]/\zeta(t)\}^2$$

where N is the dimensionality of the Brownian motion $W(t)$. Here $b_X^i(t)$ is the i -th component of the volatility vector $b_X(t)$, and $b_I^i(t)$ is the i -th component of the volatility vector $b_I(t)$.

11 Pricing zero-coupon inflation swaps and options

In this section we calculate the full expression for the price index $I(t)$: its conditional lognormality translates into closed forms (“Black type”) for ZC Inflation options. This makes the model calibration much faster. The price index dynamics in the forward measure can be used to simplify the problem by discounting via multiplication by the zero-coupon bond.

To do these analyses, we have to calculate the closed form dynamics of $I(t)$ taking into account the stochastic dynamics of its drift $m_I(t)$.

We start by integrating their T^* -forward dynamics:

$$dI(t)/I(t) = (m_I(t) - \lambda(t) \cdot s_I(t) + \sigma_P(t, T^*) \cdot s_I(t))dt + s_I(t) \cdot dW^{T^*}(t)$$

$$dm_I(t) = [a_I(t) - \lambda(t) \cdot b_I(t) + \sigma_P(t, T^*) \cdot b_I(t)]dt + b_I(t) \cdot dW^{T^*}(t)$$

We compact the notation by defining $g_1(t) = -s_I(t) \cdot (\lambda(t) - \sigma_P(t, T^*))$ and $g_2(t) = a_I(t) - b_I(t) \cdot (\lambda(t) - \sigma_P(t, T^*))$.

We notice that $g_2(t)$ is deterministic as all the quantities used to build it are deterministic. At this stage we recall that in the CTCB model the bond option volatilities are expressed as:

$$\sigma_P(t, T) = \frac{[h_x b_X(t) + h_p b_I(t)]}{(e^{\delta(t+\Omega)} - e^{\delta t})} (1 - e^{\delta(T-t)}).$$

The dynamics are therefore rewritten in a more compact form as:

$$dI(t)/I(t) = (m_I(t) + g_1(t))dt + s_I(t) \cdot dW^{T^*}(t)$$

$$dm_I(t) = g_2(t)dt + b_I(t) \cdot dW^{T^*}(t)$$

We now integrate the expression for $m_I(s)$ between times t and T :

$$\int_t^T m_I(s)ds = m_I(t)(T-t) + \int_t^T \int_t^s g_2(u)duds + \int_t^T \int_t^s b_I(u) \cdot dW^{T^*}(u)ds.$$

Applying Fubini’s theorem, we write the integral of the price index drift in a simpler form:

$$\int_t^T m_I(s)ds = m_I(t)(T-t) + \int_t^T (T-s)g_2(s)ds + \int_t^T (T-s)b_I(s) \cdot dW^{T^*}(s).$$

We can now write the normal distribution of the integral of the drift:

$$\int_t^T m_I(s)ds \sim \mathcal{N} \left(m_I(t)(T-t) + \int_t^T (T-s)g_2(s)ds, \int_t^T (T-s)^2 b_I(s) \cdot b_I(s)ds \right)$$

With the above results in mind we can now derive the expression for the price index level $I(t)$:

$$I(T) = I(t)e^{\int_t^T (m_I(t) + (T-s)g_2(s) + g_1(s) - \frac{1}{2}s_I(s) \cdot s_I(s))ds + \int_t^T ((T-s)b_I(s) + s_I(s)) \cdot dW^{T^*}(s)}$$

To achieve a lighter notation, we define:

$$g_3(s) = m_I(t) + (T-s)g_2(s) + g_1(s) - \frac{1}{2}s_I(s) \cdot s_I(s) \quad (79)$$

$$g_4(s) = (T-s)b_I(s) + s_I(s) \quad (80)$$

We note that $g_4(t)$ is deterministic. Based on the above, we obtain the following expression for the T^* -

dynamics and the terminal distribution of $I(t)$:

$$d \log I(t) = g_3(t)dt + g_4(t) \cdot dW^{T^*}(t) \quad (81)$$

$$dI(t)/I(t) = [g_3(t) + \frac{1}{2}g_4(t) \cdot g_4(t)]dt + g_4(t) \cdot dW^{T^*}(t) \quad (82)$$

$$\log \frac{I(T)}{I(t)} = \int_t^T g_3(s)ds + \int_t^T g_4(s) \cdot dW^{T^*}(s) \sim \mathcal{N} \left(\int_t^T g_3(s)ds, \int_t^T g_4(s) \cdot g_4(s)ds \right) \quad (83)$$

A similar analysis can be done for $X(t)$. We are now in a position to price zero-coupon inflation options.

Lemma 9 *The undiscounted price of a zero-coupon inflation option priced at time t with maturity T and strike K in the CTCB model is*

$$\omega e^{M+1/2V^2} N(\omega(M - (1+K)^{T-t} + V^2)/V) - \omega K N(\omega(M - (1+K)^{T-t})/V) \quad (84)$$

where $N(x) = \int_{-\infty}^x (2\pi)^{-1} e^{-\frac{s^2}{2}} ds$ and $\omega \in \{-1, 1\}$ for puts and calls respectively. Further,

$$M = \int_t^T g_3(s)ds$$

$$V^2 = \int_t^T g_4(s) \cdot g_4(s)ds.$$

Proof. Using result 67 and the distribution of the logarithm of $I(t)$ shown in 83 one obtains the above.

12 Pricing year-on-year inflation swaps and options

Here we focus our attention on year-on-year payoffs, that are model dependent. In fact, a convexity adjustment has to be introduced to take into account the co-movement of the nominal interest rate (used for discounting between times t and T_i) and the price index.

Step 1 – We obtain the dynamics of the real bond, defined as:

$$\begin{aligned} P^r(t, T) &= \mathbb{E}_t^{\mathbb{Q}}[I(T)/I(t)e^{-\int_t^T n(s)ds}] = P(t, T)\mathbb{E}_t^{\mathbb{Q}^T}[I(T)/I(t)] = \\ &= P(t, T)e^{\int_t^T g_3(s) + \frac{1}{2}g_4(s) \cdot g_4(s)ds} = P(t, T)e^{\int_t^T [m_I(s) + g_5(s)]ds} = P(t, T)e^{m_I(t)(T-t) + \int_t^T g_5(s)ds} \end{aligned}$$

where we define $g_5(s) = g_3(s) + \frac{1}{2}g_4(s) \cdot g_4(s) - m_I(t)(T-t) = g_1(s) + (T-s)g_2(s) + \frac{1}{2}g_4(s) \cdot g_4(s) - \frac{1}{2}s_I(s) \cdot s_I(s)$.

By applying Ito's lemma, taking into account the dynamics of $P(t, T)$ and $m_I(t)$, we obtain:

$$dP^r(t, T) = P^r(t, T)[(\dots)dt + \sigma_{P^r}(t, T) \cdot dW^{\mathbb{Q}}(t)]$$

where $\sigma_{P^r}(t, T) = \sigma_P(t, T) + b_I(t)(T-t)$. We are not interested in the drift component, but only in the diffusion term. This result is obtained by explicitly calculating the diffusion term:

$$\begin{aligned} &\left(\frac{\partial P^r(t, T)}{\partial P(t, T)} P(t, T) \sigma_P(t, T) + \frac{\partial P^r(t, T)}{\partial m_I(t)} b_I(t) \right) \cdot dW^{\mathbb{Q}}(t) = \\ &\left(e^{m_I(t)(T-t) + \int_t^T g_5(s)ds} P(t, T) \sigma_P(t, T) + (T-t) P(t, T) e^{m_I(t)(T-t) + \int_t^T g_5(s)ds} b_I(t) \right) \cdot dW^{\mathbb{Q}}(t) = \end{aligned}$$

$$P^r(t, T)(\sigma_P(t, T) + b_I(t)(T - t)) \cdot dW^{\mathbb{Q}}(t).$$

Step 2 – We build a T -forward martingale by entering into a zero-coupon inflation swap with notional $I(t)$ and maturity T and divide by the numeraire, i.e. the nominal bond $P(t, T)$. We recall a model-independent result that states that the present value (PV) of a zero-coupon inflation swap is the difference between the real and nominal bond of the same maturity. We get:

$$I(t)(P^r(t, T) - P(t, T))/P(t, T) = I(t)(P^r(t, T)/P(t, T) - 1).$$

We consider the quantity $I(t)P^r(t, T)/P(t, T)$, known as the forward inflation: $\hat{I}(t, T) = I(t)P^r(t, T)/P(t, T)$.

The reason why this is called forward inflation is clear if one makes the following observation:

$$P^r(t, T) = \mathbb{E}_t^{\mathbb{Q}}[I(T)/I(t)e^{-\int_t^T n(s)ds}] = P(t, T)/I(t)\mathbb{E}_t^{\mathbb{Q}^T}[I(T)]$$

Therefore one obtains:

$$\hat{I}(t, T) = I(t)P^r(t, T)/P(t, T) = \mathbb{E}_t^{\mathbb{Q}^T}[I(T)].$$

By using Ito's Lemma on $\hat{I}(t, T) = I(t)P^r(t, T)/P(t, T)$, we obtain its risk-neutral dynamics. Again, we confirm that in the T -forward dynamics the forward inflation has to be a positive martingale:

$$d\hat{I}(t, T) = \hat{I}(t, T)s_{\hat{I}}(t, T) \cdot dW^{\mathbb{Q}^T}(t)$$

where $s_{\hat{I}}(t, T)$ is determined from the other model volatilities via Ito's Lemma in the way showed below.

In particular one obtains:

$$s_{\hat{I}}(t, T) = s_I(t) + b_I(t)(T - t)$$

To see this, one applies Ito's lemma for the diffusion part of the forward inflation $\hat{I}(t, T) = I(t)P^r(t, T)/P(t, T)$:

$$\begin{aligned} & \left(\frac{\partial \hat{I}(t, T)}{\partial I(t)} I(t)s_I(t) + \frac{\partial \hat{I}(t, T)}{\partial P^r(t, T)} P^r(t, T)\sigma_{P^r}(t, T) + \frac{\partial \hat{I}(t, T)}{\partial P(t, T)} P(t, T)\sigma_P(t, T) \right) \cdot dW^{\mathbb{Q}^T}(t) = \\ & \left(\hat{I}(t, T)s_I(t) + \hat{I}(t, T)\sigma_{P^r}(t, T) - \hat{I}(t, T)\sigma_P(t, T) \right) \cdot dW^{\mathbb{Q}^T}(t) = s_I(t) + b_I(t)(T - t). \end{aligned}$$

This final step was possible thanks to the expression of the diffusion term of the real bond found in step 1.

Step 3 – By a simple application of Ito's lemma one can show that, taken some deterministic and regular functions a , b , and s , (here a is a scalar function, b and s are vectorial functions with the same dimension of the driving Brownian motion $W(t)$) if one has two SDEs define as $dX(t) = X(t)s \cdot dW(t)$ and $dY(t) = Y(t)[a dt + b \cdot dW(t)]$, the ratio $Z(t) = X(t)/Y(t)$ has dynamics $dZ(t) = Z(t)[(-a + b \cdot b - s \cdot b)dt + (s - b) \cdot dW(t)]$.

Step 4 – Similarly to what is done for the BGM model, one chooses a reference tenor T^* and changes the dynamics of the inflation forwards to the same forward measure: an example of this technique is available in Belgrade & Benhamou [4]. The dynamics of the inflation forwards were found in step 2: therefore we know explicitly the dynamics of $\hat{I}(t, T_i)$ and $\hat{I}(t, T_j)$. For example, if the reference tenor is T_i one obtains the following dynamics for the inflation forwards at tenors T_i and T_j ($T_i > T_j$):

$$d\hat{I}(t, T_i) = \hat{I}(t, T_i)s_{\hat{I}}(t, T_i) \cdot dW^{\mathbb{Q}^{T_i}}(t)$$

$$d\hat{I}(t, T_j) = \hat{I}(t, T_j)(-(\sigma_P(t, T_i) - \sigma_P(t, T_j)) \cdot s_I(t)dt + s_{\hat{I}}(t, T_j) \cdot dW^{\mathbb{Q}^{T_i}}(t))$$

Step 5 – One introduces the price index ratio process $\mathcal{I}(t, T_j, T_i) = I(t, T_i)/I(t, T_j)$: using Ito's lemma and the results at step 3 above one obtains its dynamics.

$$\begin{aligned} d\mathcal{I}(t, T_j, T_i) = \mathcal{I}(t, T_j, T_i)[& ((\sigma_P(t, T_i) - \sigma_P(t, T_j)) \cdot s_I(t, T_j) + s_{\hat{I}}(t, T_j) \cdot s_{\hat{I}}(t, T_j) - s_{\hat{I}}(t, T_i) \cdot s_{\hat{I}}(t, T_j))dt \\ & + (s_{\hat{I}}(t, T_i) - s_{\hat{I}}(t, T_j)) \cdot dW^{\mathbb{Q}^{T_i}}(t)] \end{aligned}$$

We can write the expectation of the ratio as:

$$\mathbb{E}_t^{\mathbb{Q}^{T_i}}[\mathcal{I}(t_h, T_j, T_i)] = \mathcal{I}(t, T_j, T_i) e^{\int_t^{t_h} ((\sigma_P(u, T_i) - \sigma_P(u, T_j)) \cdot s_I(u, T_j) + s_I(u, T_j) \cdot s_I(u, T_j) - s_I(u, T_i) \cdot s_I(u, T_j)) du}$$

Step 6 – Now we link the price index ratio to the year-on-year: the year-on-year forward can be expressed as an expectation of \mathcal{I} :

$$\mathbb{E}_t^{\mathbb{Q}^{T_i}}[I(T_i)/I(T_j)] = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[\hat{I}(T_i, T_i)/\hat{I}(T_j, T_j)] = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[\mathbb{E}_{T_j}^{\mathbb{Q}^{T_i}}[\hat{I}(T_i, T_i)/\hat{I}(T_j, T_j)]] = \mathbb{E}_t^{\mathbb{Q}^{T_i}}[\hat{I}(T_j, T_i)/\hat{I}(T_j, T_j)] =$$

$$\begin{aligned} \mathbb{E}_t^{\mathbb{Q}^{T_i}}[\mathcal{I}(T_j, T_j, T_i)] &= \mathcal{I}(t, T_j, T_i) e^{\int_t^{T_j} ((\sigma_P(u, T_j) - \sigma_P(u, T_i)) \cdot s_I(u) + s_I(u, T_j) \cdot s_I(u, T_j) - s_I(u, T_i) \cdot s_I(u, T_j)) du} = \\ &= \frac{\hat{I}(t, T_i)}{\hat{I}(t, T_j)} e^{\int_t^{T_j} ((\sigma_P(u, T_i) - \sigma_P(u, T_j)) \cdot s_I(u, T_j) + s_I(u, T_j) \cdot s_I(u, T_j) - s_I(u, T_i) \cdot s_I(u, T_j)) du}. \end{aligned}$$

These steps allow us to claim the following lemma.

Lemma 10 *The undiscounted price of a year-on-year inflation caplet/floorlet priced at time t with maturity T and strike K in the CTCB model is*

$$\omega e^{M+1/2V^2} N(\omega(M - (1 + K) + V^2)/V) - \omega K N(\omega(M - (1 + K))/V) \quad (85)$$

where $N(x) = \int_{-\infty}^x (2\pi)^{-1} e^{-\frac{t^2}{2}} dt$ and $\omega \in \{-1, 1\}$ for floorlets and caplets respectively. The year-on-year inflation is calculated between times T_j and T_i . Further,

$$\begin{aligned} M &= \int_t^{T_j} ((\sigma_P(u, T_i) - \sigma_P(u, T_j)) \cdot s_I(u, T_j) + s_I(u, T_j) \cdot s_I(u, T_j) - s_I(u, T_i) \cdot s_I(u, T_j)) du \\ V^2 &= \int_{T_j}^{T_i} ((\sigma_I(u, T_i) - \sigma_I(u, T_j)) \cdot (\sigma_I(u, T_i) - \sigma_I(u, T_j))) du \end{aligned}$$

Proof. Using result 67 and the distribution of the logarithm of $I(t)$ shown above one obtains the result.

13 Single currency derivatives pricing simulation

To test the results found in the previous sections, we have implemented a Monte Carlo simulator and the closed forms for zero-coupon and year-on-year inflation options. We have run 20,000 simulations over 10 years, and here we show the results, the standard error and the closed form results. This is done both for zero-coupon and year-on-year inflation options. We price caps with strikes 0, 1, 2, 3, 4, 5 percent with maturities from 1 to 10 years. We assume that the dimensionality of the driving Brownian motion is 3.

For this simulation, we assume the following set of parameters, that are constant over time: $a_X(t) = 0\%$, $a_I(t) = 0.5\%$, $b_X(t) = 0\%$, $b_I(t) = 0.3\%$, $s_X(t) = 0\%$, $s_I(t) = 0.3\%$, $\sigma_P(t, T) = 1\%$, $\lambda(t) = 0\%$, $\mu_I(0) = 0\%$; in case of vector functions, like the volatilities, we assume that the value is the same for all the 3 components. For this analysis we have only presented the parameters that are directly relevant for the pricing of inflation derivatives: a full calibration exercise will be presented in the following section.

The data show that there is good agreement between the Monte Carlo simulation and the closed forms, and that the number of simulations is high enough to control the numerical error.

The results for zero-coupon options are the following (strikes in columns, maturities in rows):

MC PV	0%	1%	2%	3%	4%	5%
1	0.00209	0.00006	0	0	0	0
2	0.00772	0.00058	0	0	0	0
3	0.01804	0.00268	0.00009	0	0	0
4	0.03359	0.00784	0.00056	0.00001	0	0
5	0.05477	0.01731	0.00219	0.00009	0	0
6	0.08208	0.03225	0.00625	0.00046	0.00002	0
7	0.11599	0.05352	0.01421	0.00167	0.00009	0
8	0.15721	0.08208	0.02753	0.00475	0.00038	0.00002
9	0.20641	0.1188	0.04775	0.01112	0.00129	0.00008
10	0.26444	0.16451	0.07619	0.02256	0.00363	0.00032

MC Error	0%	1%	2%	3%	4%	5%
1	0.00002	0	0	0	0	0
2	0.00006	0.00002	0	0	0	0
3	0.00012	0.00005	0.00001	0	0	0
4	0.00018	0.0001	0.00003	0	0	0
5	0.00027	0.00018	0.00006	0.00001	0	0
6	0.00036	0.00027	0.00012	0.00003	0	0
7	0.00047	0.00039	0.00021	0.00007	0.00002	0
8	0.0006	0.00052	0.00033	0.00014	0.00004	0.00001
9	0.00074	0.00068	0.00049	0.00024	0.00008	0.00002
10	0.00091	0.00086	0.00067	0.00038	0.00015	0.00004

PV - form	0%	1%	2%	3%	4%	5%
1	0.00212	0.00006	0	0	0	0
2	0.00779	0.00059	0.00001	0	0	0
3	0.01814	0.00271	0.00009	0	0	0
4	0.03374	0.00786	0.00057	0.00001	0	0
5	0.05502	0.0174	0.00224	0.00009	0	0
6	0.08244	0.03244	0.00631	0.00047	0.00001	0
7	0.11649	0.05391	0.01427	0.00172	0.00008	0
8	0.15781	0.08265	0.02766	0.00483	0.00038	0.00001
9	0.20712	0.11952	0.04799	0.01121	0.00133	0.00007
10	0.26534	0.16545	0.07674	0.02262	0.00371	0.00031

Difference: PV - form, MC sim.	0	1%	2%	3%	4%	5%
1	-0.00003	0	0	0	0	0
2	-0.00006	-0.00001	0	0	0	0
3	-0.0001	-0.00003	0	0	0	0
4	-0.00015	-0.00003	-0.00001	0	0	0
5	-0.00025	-0.00009	-0.00005	0	0	0
6	-0.00036	-0.00019	-0.00007	-0.00002	0	0
7	-0.0005	-0.00038	-0.00006	-0.00005	0.00001	0
8	-0.0006	-0.00057	-0.00013	-0.00008	0	0
9	-0.00071	-0.00073	-0.00025	-0.00009	-0.00004	0.00001
10	-0.0009	-0.00095	-0.00055	-0.00006	-0.00008	0.00001

The results for year-on-year options are the following:

MC PV	0%	1%	2%	3%	4%	5%
1	0.0021	0.00006	0	0	0	0
2	0.00621	0.00114	0.00007	0	0	0
3	0.01081	0.00375	0.00066	0.00005	0	0
4	0.01562	0.00736	0.00229	0.00041	0.00004	0
5	0.02055	0.01159	0.0049	0.00141	0.00025	0.00003
6	0.02549	0.01611	0.0083	0.00322	0.00087	0.00015
7	0.03051	0.0209	0.01233	0.00588	0.00212	0.00055
8	0.03558	0.02581	0.01672	0.00922	0.0041	0.00142
9	0.04066	0.0308	0.02139	0.01311	0.00681	0.00288
10	0.04566	0.03575	0.02613	0.01731	0.01005	0.00493

MC Error	0%	1%	2%	3%	4%	5%
1	0.00002	0	0	0	0	0
2	0.00004	0.00002	0	0	0	0
3	0.00006	0.00004	0.00002	0	0	0
4	0.00007	0.00006	0.00003	0.00001	0	0
5	0.00008	0.00007	0.00005	0.00003	0.00001	0
6	0.00009	0.00008	0.00007	0.00004	0.00002	0.00001
7	0.0001	0.00009	0.00008	0.00006	0.00004	0.00002
8	0.00011	0.0001	0.00009	0.00008	0.00005	0.00003
9	0.00011	0.00011	0.00011	0.00009	0.00007	0.00004
10	0.00012	0.00012	0.00012	0.0001	0.00008	0.00006

PV - form	0%	1%	2%	3%	4%	5%
1	0.00002	0	0	0	0	0
2	0.00004	0.00002	0	0	0	0
3	0.00006	0.00004	0.00002	0	0	0
4	0.00007	0.00006	0.00003	0.00001	0	0
5	0.00008	0.00007	0.00005	0.00003	0.00001	0
6	0.00009	0.00008	0.00007	0.00004	0.00002	0.00001
7	0.0001	0.00009	0.00008	0.00006	0.00004	0.00002
8	0.00011	0.0001	0.00009	0.00008	0.00005	0.00003
9	0.00011	0.00011	0.00011	0.00009	0.00007	0.00004
10	0.00012	0.00012	0.00012	0.0001	0.00008	0.00006

Difference: PV - form, MC sim.	0%	1%	2%	3%	4%	5%
1	0.00003	0	0	0	0	0
2	0.00068	0.00065	0.00006	0	0	0
3	0.00036	0.0007	0.00034	0.00004	0	0
4	0.00019	0.00051	0.00053	0.0002	0.00003	0
5	0.00011	0.00034	0.0005	0.00035	0.00011	0.00002
6	0	0.00016	0.00036	0.00041	0.00022	0.00006
7	-0.00002	0.00008	0.00028	0.0004	0.00031	0.00014
8	-0.00002	0.00005	0.00018	0.00033	0.00035	0.00024
9	0	0.00004	0.00014	0.00029	0.00038	0.00032
10	-0.00006	-0.00004	0.00004	0.00017	0.0003	0.00033

14 Extension to the open economy

This setting also allows to price inflation derivatives that are struck in a different currency. To do this, one simply defines the quantities seen in the previous sections also for the foreign economy and then introduces a domestic risk-neutral process for the FX rate $\{Y(t)\}_{t \geq 0}$. One assumes that the foreign economy works in a similar way, that there is a foreign central bank and that there is a liquidity relationship in the foreign bond market between foreign bond prices and foreign money supply. In particular, all parameters for the foreign economy variables are the same used in the domestic one, with an index f .

The dynamics of the foreign assets and other quantities are:

$$dX^f(t)/X^f(t) = (m_{X^f}(t) - \lambda(t) \cdot s_{X^f}(t))dt + s_I(t) \cdot dW^{\mathbb{Q}^f}(t)$$

$$dI^f(t)/I^f(t) = (m_{I^f}(t) - \lambda(t) \cdot s_{I^f}(t))dt + s_I(t) \cdot dW^{\mathbb{Q}^f}(t)$$

$$dm_{X^f}(t) = (a_{X^f}(t) - \lambda(t) \cdot b_{X^f}(t))dt + b_{X^f}(t) \cdot dW^{\mathbb{Q}^f}(t)$$

$$dm_{I^f}(t) = (a_{I^f}(t) - \lambda(t) \cdot b_{I^f}(t))dt + b_{I^f}(t) \cdot dW^{\mathbb{Q}^f}(t)$$

$$dP^f(t, T)/P^f(t, T) = n^f(t)dt + \sigma_{P^f}(t, T) \cdot dW^{\mathbb{Q}^f}(t)$$

$$dn^f(t) = [f_1^f(t) - f_2^f(t)n^f(t) - \lambda(t) \cdot \sigma_{n^f}(t)]dt + \sigma_{n^f}(t) \cdot dW^{\mathbb{Q}^f}(t)$$

$$dY(t)/Y(t) = (n(t) - n^f(t))dt + s_Y(t) \cdot dW^{\mathbb{Q}}(t)$$

We are assuming that the same Brownian motion drives both the domestic and the foreign economy. In particular the parameters for the foreign short rate dynamics are defined in the same way the domestic ones were defined:

$$f_2^f(t) = [Z^f(t + \Omega) - Z^f(t)]/\zeta^f(t) \quad (86)$$

$$f_1^f(t) = [-h_p^f a_{I^f}(t) - h_x^f a_{X^f}(t)]/\zeta^f(t) \quad (87)$$

$$\sigma_n^f(t) = [-h_x^f b_{X^f}(t) - h_p^f b_{I^f}(t)]/\zeta^f(t) \quad (88)$$

By changing the numeraire in the domestic economy from $B(t)$ to $Y(t)B^f(t)$, one achieves the domestic risk neutral dynamics for the foreign economic variables. This translates into a change of drift of $s_{(\cdot)}(t) \cdot s_Y(t)$, where $s_{(\cdot)}(t)$ is the Brownian volatility for a generic model variable.

15 Uncertain-parameters extension: modelling the inflation smile

The model presented so far gets its randomness from an n -dimensional Brownian motion $W(t)$. In principle, one can extend the theory proposed so far to the Merton jump-diffusion case, which could add great flexibility especially to model inflation skew. Here we show that the Merton equation can be obtained in the framework

proposed above if one assumes that the model has uncertain parameters. An uncertain-parameters model is a model whose parameters can take random values that are known at inception. Normally one assumes that there is a finite number of possible levels for the parameters and that the parameter set is determined one instant before the process starts. The distributions of the state variables are mixtures of distributions. For an introduction to these models one can see Brigo & Mercurio [7].

In the Merton Model the source of randomness is the process $\Lambda(t) = s_I(t)W(t) + \sum_{i=1}^{N(t)}(J_i - 1)$, where:

1. $N(t)$ is a Poisson process with intensity h independent from $W(t)$ and J_1, J_2, \dots
2. The log of the jump size J has a normal distribution with mean μ_J and variance $(\delta_J)^2$. Therefore the expected jump size is $\mathbb{E}[J - 1] = e^{\mu_J + \frac{1}{2}(\delta_J)^2} - 1 = k$. The log of the jump size is also independent from the Brownian motion $W(t)$.
3. The drift of the process has been adjusted to take into account the compensator: $\mu_I - hk$.

For simplicity, here we consider a one-dimensional source of randomness. The equation governing the evolution of the price index would read:

$$dI(t) = I(t)[(\mu_I - hk)dt + d\Lambda^{\mathbb{P}}(t)] = I(t)[(\mu_I - hks_I(t))dt + s_I(t)dW^{\mathbb{P}}(t) + (J - 1)dN^{\mathbb{P}}(t)]$$

As Merton has showed, because the distribution of J is lognormal, the distribution of $\log[I(T)/I(t_0)]$ is still normal conditional to the event $\{N(T) = n\}$.

Therefore one can regard such model as an uncertain-parameters model, where, with probability $p(N(T) = n) = e^{-hT}(hT)^n/n!$, the SDE for $I(t)$ is:

$$dI(t) = I(t)[(\mu_{(I)} - hks_I(t) + n(\mu_J + \frac{1}{2}\delta_J^2)/(T - t_0))dt + ((s_I(t))^2 + n(\delta_J)^2/(T - t_0))^{\frac{1}{2}}dW^{\mathbb{P}}(t)]$$

Therefore the theory developed so far for the Brownian case is extended to the Merton case by making some assumptions regarding uncertain model parameters.

16 Model calibration and applications

Here we propose a strategy to calibrate the CTCB model to market observables by finding suitable model parameters, and show some practical applications. Two main advantages become apparent: firstly, the CTCB calibration process is separable and model is analytically tractable and secondly, because it is based on economic theory, one can run some economic stress scenarios and obtain the answers directly from the model itself, without having to make assumptions on how an economic shock would impact on financial quantities such as inflation and rates (this is an answer provided by the model itself).

17 At-the-money calibration strategy

We are calibrating the model at time $t_0 = 0$: we are still making the assumption that the market observables are continuous functions of the maturity, to keep the notation light (this assumption will be removed in the next sections). When calibrating vector parameters, like $b_I(t)$ or $s_I(t)$ to name a few, we do not make any assumptions regarding how the total quantity needed to calibrate the model is split across the single components: this topic will be analysed later.

17.1 Calibration steps: a first strategy

1. One makes explicit the assumptions on the structural parameters of the model, namely the reaction function parameters h_x and h_p , the reaction function targets \bar{x} and \bar{p} , the liquidity horizon of the central bank Ω , and the shape of the function $Z(T)$ by choosing the parameter $\delta > 0$. In practice,

these parameters are to be seen not as an target for the calibration, but as some input from economic research that is expected to stay constant over time. The GDP volatility $s_X(t)$ can be seen as an input of the model, and therefore estimated based on some historic data.

2. Known the parameter $\delta > 0$, one can write $Z(T) = e^{\delta T}$ and $\zeta(t) = \int_t^{t+\Omega} Z(T)dt = (\delta^{-1})(e^{\delta(t+\Omega)} - e^{\delta t})$. Further one finds the function $\beta(T) = B(0, T) = e^{-\delta T}$: we remind the reader that this function is the one used in the Ansatz $P(t, T) = A(t, T)e^{-n(t)B(t, T)}$ that characterises the nominal bond prices $P(t, T)$ as a function of the short rate $n(t)$. Once $B(0, t)$ is found, from the market bond prices $P(0, T)$ and the market quote for $n(0)$ one can deduce the function $A(0, T) = P(0, T)/e^{-n(0)e^{-\delta T}}$. One should remember that the functions $A(t, T)$ and $B(t, T)$ are not core functions of the CTCB model, but are only relevant to its dual Hull-White model. Finally, if needed one can get the function $B(t, T) = [B(0, T) - B(0, t)]/(\partial B(0, t)/\partial t) = (e^{-\delta T} - e^{-\delta t})/(-\delta e^{-\delta t}) = (\delta^{-1})(1 - e^{-\delta(T-t)})$: this is a standard result in the Hull-White model.
3. By exploiting the fact that a CTCB model implies an equivalent (dual) Hull-White model for the short rate $n(t)$, one can immediately calculate the mean reversion speed $a(t)$: as proved in the previous section, the parametrisation $Z(T) = e^{\delta T}$ implies that the mean reversion speed is constant and equivalent to δ ; in fact we recall that

$$a(t) = [Z(t + \Omega) - Z(t)]/\zeta(t) = \frac{Z(t + \Omega) - Z(t)}{\int_t^{t+\Omega} Z(T)dT} = \delta. \quad (89)$$

We notice that we are still missing the short rate volatility $\sigma_n(t)$ and the market price of risk $\lambda(t)$ to get the mean reversion level function $\theta(t)$. This function will be found in the following steps.

4. One takes from the market quotes of at-the-money caps and floors: from these it is straightforward to get the single at-the-money caplets and floorlets. These are sensitive to the interest rate volatility, and can be used to calibrate some CTCB model volatilities: we recall that at time t the price of a caplet with strike K on the Libor between times T_{i-1} and T_i , is equivalent to the price of a put option with expiry T_{i-1} on a zero-coupon bond with maturity $T_i > T_{i-1}$. The price of such option can be obtained in closed form via a Black-type formula in the Hull-White model, where the total variance used for pricing is:

$$V^2(t, T_{i-1}, T_i) = [\beta(T_i) - \beta(T_{i-1})]^2 \int_t^{T_{i-1}} \left[\frac{\sigma_n^*(u)}{\beta'(u)} \right]^2 du. \quad (90)$$

Because $\beta(T) = B(0, T) = e^{-\delta T}$ and $\beta'(T) = \partial B(0, T)/\partial T = -\delta e^{-\delta T}$, the above is written as:

$$V^2(t, T_{i-1}, T_i) = [e^{-\delta T_i} - e^{-\delta T_{i-1}}]^2 \int_t^{T_{i-1}} \left[\frac{\sigma_n^*(u)}{-\delta e^{-\delta u}} \right]^2 du \quad (91)$$

Finally, we recall that in the equivalent Hull-White model the short rate volatility was expressed as:

$$\sigma_n(t) = [-h_x b_X(t) - h_p b_I(t)]/\zeta(t) = -\delta[h_x b_X(t) + h_p b_I(t)]/(e^{\delta(t+\Omega)} - e^{\delta t}). \quad (92)$$

One should refer to 60 to see how one moves from the scalar original Hull-White volatility $\sigma_n^*(t)$ to the vector volatility in the CTCB model, denoted as $\sigma_n(t)$.

Therefore, at the end of this step we have fully calibrated the CTCB model to the nominal term structure and at-the-money caps-floors volatilities, and found the economic expectation volatility functions $b_X(t)$ and $b_I(t)$: one chooses these functions freely to ensure that the model at-the-money cap-floors prices match the ones observed in the market. If one wanted to use the CTCB model to price nominal rate derivatives, the calibration process could be ended here. Alternatively, one can specify the function $b_X(t)$ based on historic data and only calibrate $b_I(t)$.

5. A first consequence of the above result is that, exploiting the standard result $\sigma_P(t, T) = -\sigma_n(t)B(t, T)$ in the Hull-White model, we can write explicitly the bond volatilities: these will be needed either if one needs to simulate Libor rates ($F(t, T_{i-1}, T_i) = (P(t, T_{i-1})/P(t, T_i) - 1)/(T_i - T_{i-1})$) or when building the drift adjustment to move to the T^* -forward measure. Making all dependencies explicit we write

$$\sigma_P(t, T) = -\delta \frac{[h_x b_X(t) + h_p b_I(t)]}{(e^{\delta(t+\Omega)} - e^{\delta t})} (\delta^{-1})(1 - e^{-\delta(T-t)}) = \frac{[h_x b_X(t) + h_p b_I(t)]}{(e^{\delta(t+\Omega)} - e^{\delta t})} (1 - e^{-\delta(T-t)}). \quad (93)$$

6. The function $A(t, T)$ can now be made explicit using a standard Hull-White result:

$$\log A(t, T) = \log A(0, T) - \log A(0, t) - B(t, T) \frac{\partial \log A(0, t)}{\partial t} - \frac{1}{2} \left[B(t, T) \frac{\partial B(0, t)}{\partial t} \right]^2 \int_0^t \left[\frac{\sigma_n(s)}{\frac{\partial B(0, s)}{\partial s}} \right]^2 ds. \quad (94)$$

7. We now calibrate to the inflation volatilities implied by the market. We recall that the total variance of the quantity $\log(I(T)/I(0))$ is $\int_0^T g_4(s) \cdot g_4(s) ds$ where $g_4(t) = (T-t)b_I(t) + s_I(t)$. Therefore one can find the function $s_I(t)$, under the constraint that we know already the function $b_I(t)$, using the closed forms for inflation zero-coupon options found in the previous section.

8. At this stage one has enough information to calibrate the model to the inflation breakeven strikes from zero-coupon inflation swaps, remembering that the expectation of the quantity $\log(I(T)/I(0))$ is $\int_0^T g_3(s) ds$ where we recall the definitions of $g_1(t) = -s_I(t) \cdot (\lambda(t) - s_P(t, T^*))$, $g_2(t) = a_I(t) - b_I(t) \cdot (\lambda(t) - s_P(t, T^*))$, and $g_3(t) = m_I(0) + (T-t)g_2(t) + g_1(t) - \frac{1}{2} s_I(t) \cdot s_I(t)$. We remind the reader that these results were found under the T^* -forward measure. Therefore we have now found the market prices of risk function $\lambda(t)$ and the inflation expectation drift function $a_I(t)$. Alternatively, one can specify the market price of risk $\lambda(t)$ based on historic data and calibrate only $a_I(t)$. The former alternative is well-suited for relative value analysis, i.e. the trader, based on a view of the economy and the observed market prices, backs out the market prices of risks implied by market prices, and gauges the illiquidity spots or the inconsistencies between the market participants' risk preferences. The latter is more suited to replicate market prices, i.e. the trader makes an assumption on the market risk aversion and backs out the implied paths for the price index and GDP growth expectations.

9. We can write the Hull-White equivalent mean reversion level by exploiting the standard result:

$$\theta(t) = \lambda(t)\sigma_n(t) - \delta \frac{\partial \log A(0, t)}{\partial t} - \frac{\partial^2 \log A(0, t)}{\partial t^2} + \left[\frac{\partial B(0, t)}{\partial t} \right]^2 \int_0^t \left[\frac{\sigma_n(s)}{\frac{\partial B(0, s)}{\partial s}} \right]^2 ds. \quad (95)$$

This result is used to find the growth expectation drift function $a_X(t)$ remembering that the mean reversion level in the Hull-White equivalent model was given by $\theta(t) = [-h_p a_I(t) - h_x a_X(t)]/\zeta(t)$.

10. The previous two points can be compacted into one, if one assumes to know the expectation drifts $a_X(t)$ and $a_I(t)$ and therefore calibrates only the market price of risk $\lambda(t)$.
11. Finally, one recalls the volatility condition 41 to calculate $s_M(t) = h_p s_I(t) + h_x s_X(t)$: these volatilities may be needed to run a full simulation of the model but are not needed to price derivatives.

17.2 Calibration steps: an alternative strategy

Here we propose a minor change to the calibration strategy seen above that can be introduced in order to ensure full calibration of the model. In fact, we are calibrating the functions $b_X(t)$ and $b_I(t)$ first, based on the market prices of nominal caps and floors (step 4): in a second step (step 7) we find the function $s_I(t)$ that calibrates the market prices of inflation zero-coupon options. In this step there can be a problem, given

that the total variance is $\int_0^T g_4(s) \cdot g_4(s) ds$, where $g_4(t) = (T - t)b_I(t) + s_I(t)$: the function $s_I(t)$ in some cases can only increase the total variance given $b_I(t)$, and, further, the function $b_I(t)$ is multiplied by $T - t$, which can lead to excessive implied variance at long maturities. In practice, in some cases the model may not be able to calibrate to inflation zero-coupon options, because it can not reduce the implied variance below a certain threshold. If inflation volatilities are too low, full calibration may not be achieved.

To overcome this problem, we suggest to calibrate the functions $b_I(t)$ and $s_I(t)$ to inflation zero-coupon options across all maturities as a first step, and then to use the function $b_X(t)$ to calibrate the nominal caps and floors: the advantage will be that calibration will be guaranteed in both inflation and caps and floors volatilities, however the trader will not be able to freely mark the output expectation volatilities $b_X(t)$, which was possible in the approach proposed originally. We do not have an explicit preference for either approach: the choice will depend on whether one wants to control the output expectation volatilities $b_X(t)$ or guarantee a full calibration to option prices.

17.3 Variance split and calibration to correlations

In the steps of the previous section we found some model volatilities, namely $b_I(t)$, $b_X(t)$, $s_I(t)$, and $s_X(t)$. Because these processes are multidimensional with dimension n , given some option prices one wants to calibrate to, there are multiple ways to split the total variance that one is targeting for a given processes into its components. We regard this fact as an opportunity to calibrate to a correlation structure that the trader can choose.

Let us take the model volatility process $b_I(t)$: all we say for it can be exactly extended to the remaining three processes. We introduce some weights, called $w_{b_I(t)}^i$ with $i = 1, 2, \dots, n$ and such that $\sum_{i=1}^n [w_{b_I(t)}^i]^2 = 1$. In theory one can write $v_{b_I(t)} = \sum_{i=1}^n [b_I^i(t)]^2$, where $b_I^i(t)$ is the i -th component of $b_I(t)$: in practice the calibration process will only yield $v_{b_I(t)}$. One will define $[b_I^i(t)]^2 = v_{b_I(t)} [w_{b_I(t)}^i]^2$ and be guaranteed that the total variance will be split according to some pre-defined weights: this is done for all four model volatilities, yielding the total variances $v_{b_I(t)}$, $v_{b_X(t)}$, $v_{s_I(t)}$, and $v_{s_X(t)}$, assuming that one knows the weights $w_{b_I(t)}^i$, $w_{b_X(t)}^i$, $w_{s_I(t)}^i$, and $w_{s_X(t)}^i$.

These four sets of weights can be determined in a way to target a given correlation level. Let us define the list of the variables for which we want to impose a correlation structure. A list could be made of the changes in the short rate $dn(t)$, relative changes in the price index $dI(t)/I(t)$. Finally one may be interested to impose a correlation structure that includes the relative changes of the real GDP $dX(t)/X(t)$. We assume we know the market-implied 3×3 correlation matrix.

We want to find the weights $w_{b_I(t)}^i$, $w_{b_X(t)}^i$, $w_{s_I(t)}^i$, and $w_{s_X(t)}^i$ that allow the instantaneous model correlations to be as close as possible to the market-implied correlations: clearly there is a trade-off between the accuracy of this fit and the dimensionality n of the Brownian motion $\{W(t)\}_{t \geq 0}$. A high enough dimensionality n can ensure a perfect fit, however this would make the model overparametrised and difficult to manage. The accuracy is measured as the square difference between the market implied correlation $\rho_{a(t), b(t)}^{MKT}(t)$ between the generic variables $a(t)$ and $b(t)$ and the model correlations $\rho_{a(t), b(t)}^{MOD}(t)$: here $a(t) \in \mathcal{V} = \{dn(t), dI(t)/I(t), dX(t)/X(t)\}$ and $b(t) \in \mathcal{V}$.

We know the model volatility functions for the 3 variables in closed form from the previous section, for the short rate change, for the price index relative change, and for the output relative change respectively:

$$\begin{aligned} ModelVol(n(t)) &= \frac{-\delta[h_x b_X(t) + h_p b_I(t)]}{(e^{\delta(t+\Omega)} - e^{\delta t})} \\ ModelVol(I(t)) &= (T - t)b_I(t) + s_I(t) \\ ModelVol(X(t)) &= (T - t)b_X(t) + s_X(t). \end{aligned}$$

In general, for two generic driftless scalar real processes $Y(t)$ and $Z(t)$, four real constants a, b, c, f , two n -

dimensional vector volatility deterministic real processes $\{s_1(t)\}_{t \geq 0}$ and $\{s_2(t)\}_{t \geq 0}$, and for an n -dimensional Brownian motion $\{W(t)\}_{t \geq 0}$ with independent components, we can assume the following dynamic equations:

$$\begin{aligned} dY(t) &= (as_1(t) + bs_2(t)) \cdot dW(t) \\ dZ(t) &= (cs_1(t) + fs_2(t)) \cdot dW(t) \end{aligned}$$

We drop the time dependency to make the notation lighter and write the above as a sum of component-by-component products. The Brownian motion differential components are denoted by dW_i , while the single volatility components are denoted by s_1^i and s_2^i :

$$\begin{aligned} dY &= a \sum_{i=1}^n s_1^i dW_i + b \sum_{i=1}^n s_2^i dW_i \\ dZ &= c \sum_{i=1}^n s_1^i dW_i + f \sum_{i=1}^n s_2^i dW_i \end{aligned}$$

We substitute the single components using the total variance technique proposed above, writing $[s_1^i]^2 = v_1[w_1^i]^2$ and $[s_2^i]^2 = v_2[w_2^i]^2$:

$$\begin{aligned} dY &= a \sum_{i=1}^n v_1^{\frac{1}{2}}[w_1^i] dW_i + b \sum_{i=1}^n v_2^{\frac{1}{2}}[w_2^i] dW_i \\ dZ &= c \sum_{i=1}^n v_1^{\frac{1}{2}}[w_1^i] dW_i + f \sum_{i=1}^n v_2^{\frac{1}{2}}[w_2^i] dW_i \end{aligned}$$

Now we want to write the instantaneous correlation between $dY(t)$ and $dZ(t)$, written as

$$\rho_{dY(t), dZ(t)}(t) = \frac{\langle dY(t), dZ(t) \rangle}{[\langle dY(t), dY(t) \rangle \langle dZ(t), dZ(t) \rangle]^{\frac{1}{2}}}.$$

By doing the calculations and thanks to the independence of the components of the Brownian motions, one gets:

$$\begin{aligned} \langle dY, dZ \rangle &= \left[a \sum_{i=1}^n v_1^{\frac{1}{2}}[w_1^i] dW_i + b \sum_{i=1}^n v_2^{\frac{1}{2}}[w_2^i] dW_i \right] \left[c \sum_{i=1}^n v_1^{\frac{1}{2}}[w_1^i] dW_i + f \sum_{i=1}^n v_2^{\frac{1}{2}}[w_2^i] dW_i \right] = \\ &= (af + bc) v_1^{\frac{1}{2}} v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i dt + ac v_1 \sum_{i=1}^n [w_1^i]^2 dt + bf v_2 \sum_{i=1}^n [w_2^i]^2 dt = \left\{ (af + bc) v_1^{\frac{1}{2}} v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i + ac v_1 + bf v_2 \right\} dt. \end{aligned}$$

For the denominator terms one writes similarly:

$$\langle dY, dY \rangle = \left[a \sum_{i=1}^n v_1^{\frac{1}{2}}[w_1^i] dW_i + b \sum_{i=1}^n v_2^{\frac{1}{2}}[w_2^i] dW_i \right] \left[a \sum_{i=1}^n v_1^{\frac{1}{2}}[w_1^i] dW_i + b \sum_{i=1}^n v_2^{\frac{1}{2}}[w_2^i] dW_i \right] =$$

$$\begin{aligned}
& (a^2 v_1 + b^2 v_2)dt + 2abv_1^{\frac{1}{2}}v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i dt \\
\langle dZ, dZ \rangle &= \left[c \sum_{i=1}^n v_1^{\frac{1}{2}} [w_1^i] dW_i + f \sum_{i=1}^n v_2^{\frac{1}{2}} [w_2^i] dW_i \right] \left[c \sum_{i=1}^n v_1^{\frac{1}{2}} [w_1^i] dW_i + f \sum_{i=1}^n v_2^{\frac{1}{2}} [w_2^i] dW_i \right] = \\
& (c^2 v_1 + f^2 v_2)dt + 2cfv_1^{\frac{1}{2}}v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i dt
\end{aligned}$$

One can therefore write:

$$\rho_{dY(t), dZ(t)}(t) = \frac{(af + bc)v_1^{\frac{1}{2}}v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i + acv_1 + bf v_2}{[(a^2 v_1 + b^2 v_2) + 2abv_1^{\frac{1}{2}}v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i]^{\frac{1}{2}} [(c^2 v_1 + f^2 v_2) + 2cfv_1^{\frac{1}{2}}v_2^{\frac{1}{2}} \sum_{i=1}^n w_1^i w_2^i]^{\frac{1}{2}}}.$$

It is clear that the above generic parametrisation is a slight simplification of the format of all SDEs in the CTCB model, and therefore can be used as a general framework (the simplification is that in the above example for clarity we have assumed only two model volatility functions $s_1(t)$ and $s_2(t)$, while the CTCB has four, namely $b_I(t)$, $b_X(t)$, $s_I(t)$, and $s_X(t)$). Here we notice that, known some model parameters a, b, c, f and the total variances v_1 and v_2 from the previous calibration step, one can choose the weights w_1^i and w_2^i to target a specific correlation level.

For example, to get $\rho_{n(t), I(t)}^{MKT}(t)$, one writes:

$$\rho_{n(t), I(t)}^{MKT}(t) = \frac{\langle \sigma_n(t) \cdot dW(t), [(T-t)b_I(t) + s_I(t, T)] \cdot dW(t) \rangle}{[\langle \sigma_n(t) \cdot dW(t), \langle \sigma_n(t) \cdot dW(t) \rangle \rangle \langle [(T-t)b_I(t) + s_I(t, T)] \cdot dW(t), [(T-t)b_I(t) + s_I(t, T)] \cdot dW(t) \rangle]^{\frac{1}{2}}}$$

By doing the calculations one gets to the final result.

This example shows that all model correlations can be computed in closed form as a function of the known model parameter and the unknown model volatilities weights $w_{b_I(t)}^i$, $w_{b_X(t)}^i$, $w_{s_I(t)}^i$, and $w_{s_X(t)}^i$.

The non-linear optimisation problem can be formalised as follows:

$$\min \sum_{a(t) \in \mathcal{V}} \sum_{b(t) \in \mathcal{V}} [\rho_{a(t), b(t)}^{MKT}(t, w_{b_I(t)}^i, w_{b_X(t)}^i, w_{s_I(t)}^i, w_{s_X(t)}^i) - \rho_{a(t), b(t)}^{MOD}(t)]^2 \quad (96)$$

under the constraints: $\sum_{i=1}^n [w_{b_I(t)}^i]^2 = 1$, $\sum_{i=1}^n [w_{b_X(t)}^i]^2 = 1$, $\sum_{i=1}^n [w_{s_I(t)}^i]^2 = 1$, $\sum_{i=1}^n [w_{s_X(t)}^i]^2 = 1$.

18 At-the-money calibration results

18.1 Technical assumptions

Some operational assumptions are made to deal with the data, and they are not to be considered part of the core model construction; however they need to be made explicit here. In general, when making choices, we assume we want to maximise the calibration accuracy for pricing purposes, as a market maker would do.

1. We assume that all model functions $b_I(t)$, $s_I(t)$, $a_I(t)$, $b_X(t)$, $s_X(t)$, and $a_X(t)$ as step functions, where the discontinuities happen at the quoted maturities.
2. We linearly interpolate the market observables at equally spaced times steps, where the time interval is one year. The market observables are the nominal interest curve, the inflation zero-coupon curve, the prices of at-the-money caplets, and the prices of at-the-money zero-coupon inflation options.
3. At-the-money caplets are not directly traded in the market, but are recovered as differences between the PV of the at-the-money caps of two maturities.

4. The prices of zero-coupon inflation options are not quoted for at-the-money strikes but for fixed strikes, therefore a second linear interpolation across strikes is done for each maturity.
5. We assume that the market prices of risk are constant and equal to zero for all components: therefore one backs out the risk-neutral paths for the expected inflation and growth rate.
6. We calibrate inflation options first and then nominal caplets, by keeping the function $b_X(t)$ constant. Therefore we use the “alternative strategy” detailed in the previous sections.
7. The dimensionality of the driving Brownian motion is 3. The choice appears to be a good compromise between model simplicity and calibration flexibility.
8. For the zero-finding routine, we used Newton’s method with maximum 5,000 iterations and absolute price difference tolerance of 0.00000001.
9. The integrals such as the ones in 91 that have to be solved to calculate prices of caplets are calculated using the rectangles method with a time interval of 0.01 years.
10. The weights used in the correlation targeting step to allocate the variance between the different components of the noise source are assumed to be constant over time.
11. The correlations assumed are: -60% for interest rates/inflation, -60% for interest rates/growth, and 70% for inflation/growth; they are chosen by following standard economic theory. Higher interest rates reduce growth and inflation. Higher growth normally brings about higher inflation.

18.2 Economic assumptions

We made the following assumptions regarding the static model parameters.

Parameter	Level
δ	0.05
h_P	1.75
h_X	2.5
\bar{p}	2%
\bar{x}	2%
Ω	5

This model parametrisation is certainly subjective, however it reflects our view that the European Central Bank (ECB) under governor Draghi is attaching more importance to growth than inflation (therefore $h_X > h_P$) in the last years. The ECB’s official inflation target is 2%, and it is consensus between economists that the long term growth rate of a developed economy should be around 2%. Finally, up to 2012 the ECB had no tradition to doing quantitative easing on long maturities (like, for example, the FED): therefore we cap the maturity of the instruments used for monetary policy to 5 years ($\Omega = 5$). The choice of δ has been made as follows: in the Hull-White model, this parameter is the product between the long-term equilibrium level for the short interest rate and the adjustment speed. Because interest rates are at historic lows, we assume a much higher equilibrium level at 4%: further, an acceptable adjustment speed is 1.25, therefore yielding $\delta = 0.05$. To ensure the stability of the calibration, these parameters have been shocked and the model was recalibrated satisfactorily.

18.3 Market data

We calibrate the model to the European inflation market as of 7th December 2012 (data show below).

Maturity (years)	Nominal IR	Inflation ZC B/E	ATM Caplet PV	ATM ZC Infl. Option PV
1	0.0022	0.0152	0.0007	0.0039
2	0.0026	0.016	0.0017	0.0086
3	0.0045	0.0163	0.0044	0.0147
4	0.0063	0.0166	0.0055	0.0234
5	0.0081	0.017	0.0076	0.0317
6	0.01	0.0173	0.0094	0.0402
7	0.0118	0.0176	0.0108	0.0483
8	0.0136	0.0182	0.0119	0.0594
9	0.0152	0.0189	0.0127	0.0696
10	0.0168	0.0195	0.0134	0.079

18.4 Correlation targeting

Correlation targeting has been achieved by finding the variance weights $w_{b_I(t)}^i$, $w_{b_X(t)}^i$, $w_{s_I(t)}^i$, and $w_{s_X(t)}^i$ under constraints. Given that the dimension of the Brownian motion is 3, we have to find 8 weights (2 weights for each function, given that the third is calculated from the request that the sum of their squares has to be 1). We assumed that the weights are constant over time. The result of the numerical optimisation is:

	1	2	3
$w_{b_I(t)}^i$	0.20285	0.13219	0.97024
$w_{b_X(t)}^i$	-0.95101	-0.02865	0.30781
$w_{s_I(t)}^i$	0.14035	0.10000	0.98503
$w_{s_X(t)}^i$	0.85195	-0.07168	0.51868

Interestingly the weights for $b_I(t)$ and $s_I(t)$ have the same sign across the 3 components, which seems consistent with the original idea of the DSGE macroeconomic model, i.e. that inflation depends heavily on its expectations. Instead, the first component shows different signs for $b_X(t)$ and $s_X(t)$, which is consistent with the idea of productivity shocks, i.e. that the growth expectations can differ from realised growth rates.

18.5 Results

The following model parameters have been found for the price index processes:

Maturity (years)	$b_I(t)$			$s_I(t)$			$a_I(t)$
1	-0.000269	0.000314	0.000911	0.000273	0.005481	0.007732	-0.000059
2	-0.000269	0.000314	0.000911	0.000512	0.010292	0.014518	0.001543
3	-0.000269	0.000314	0.000911	0.000783	0.015726	0.022182	-0.000174
4	-0.000269	0.000314	0.000911	0.001167	0.023437	0.03306	0.000697
5	-0.000269	0.000314	0.000911	0.001317	0.026467	0.037333	0.00026
6	-0.000269	0.000314	0.000911	0.001461	0.029347	0.041396	0.00016
7	-0.000269	0.000314	0.000911	0.001517	0.030486	0.043003	0.00043
8	-0.000269	0.000314	0.000911	0.001919	0.038553	0.054382	0.003361
9	-0.000269	0.000314	0.000911	0.001895	0.038063	0.053691	0.000545
10	-0.000269	0.000314	0.000911	0.001803	0.036218	0.051088	0.00151

The following model parameters have been found for the GDP processes:

Maturity (years)	$b_X(t)$			$s_X(t)$			$a_X(t)$
1	-0.003463	-0.000807	0.001562	0.009875	0.000464	0.001507	-0.009337
2	-0.008657	-0.002016	0.003905	0.009875	0.000464	0.001507	-0.015211
3	-0.024641	-0.005739	0.011115	0.009875	0.000464	0.001507	-0.008325
4	-0.022408	-0.005219	0.010107	0.009875	0.000464	0.001507	-0.013803
5	-0.038067	-0.008867	0.01717	0.009875	0.000464	0.001507	-0.013651
6	-0.045838	-0.010677	0.020675	0.009875	0.000464	0.001507	-0.013985
7	-0.049767	-0.011592	0.022448	0.009875	0.000464	0.001507	-0.013004
8	-0.053907	-0.012556	0.024315	0.009875	0.000464	0.001507	-0.015206
9	-0.057733	-0.013353	0.025859	0.009875	0.000464	0.001507	-0.012411
10	-0.057733	-0.013353	0.025859	0.009875	0.000464	0.001507	-0.009437

In all cases the absolute calibration error has been below the wanted threshold of 0.0000001.

19 Applications

19.1 Derivatives risk as a function of the central bank reaction function

We price a 2% zero-coupon inflation cap with 10 years maturity and 1 EUR notional in the CTCB model and then shock the central bank reaction function parameters. In this way we assess the impact of a sudden (and not hedgeable) change in the central bank reaction function (or, more practically, of a new president of the central bank who may have different views compared to the current one).

In particular we find that inflation delta (defined as the change in PV when the inflation curve is shifted up by 1 basis point) is not sensitive to the central bank reaction function parameters (we shock separately the parameters h_P and h_X by 0.5 and in both cases the inflation delta stays at 0.04447): this is expected as the sensitivity of an inflation claim to inflation should mainly depend on the inflation level and not by the central bank reaction function.

19.2 Inflation book macro-hedging in the CTCB model

Let us assume that an investment bank has sold a low strike inflation floor, which is a popular hedge against deflation: for example a macro hedge fund may want to buy protection against a deflation scenario given some views. This trade would probably make good margin for the bank, given the relative low liquidity of low strike inflation options, however this would expose the bank to a significant downside risk that is difficult to recycle. An option for the bank would be to buy a nominal interest rates floor, given that this market is much more liquid than the inflation options market: the idea would be that in a low inflation environment interest rates would go down, therefore making money on the long nominal interest rates hedge while losing on the short inflation client trade. Investment banks use different models to price nominal rates and inflation trades, and the decision on the amount of nominal hedge to buy to offset the short inflation position is taken in a very informal way. This can lead to significant losses and basis risk. We argue that one of the advantages of the CTCB model is that it offers a global representation of the economy and allows consistent pricing of interest rates and inflation trades with no ambiguity: this is because this hedging problem boils down to how the central bank can affect the nominal yield curve given a deflationary scenario.

For example, one can use the calibrated CTCB model to run a Monte Carlo simulation over the maturity of the inflation client trade. One selects the paths where inflation has gone down enough for the short client trade to be in the money, and obtain a conditional distribution for the forward Libor rates given the inflation decrease: by pricing nominal floors in these scenarios, the trader can assess what nominal rates strikes are best used to hedge a deflationary scenario, choose the cheapest strikes, and, most importantly, calculate a scenario-driven hedge ratio. We stress that this example is not a pricing application, and therefore there will be some profit and loss volatility during the life of the trade, as we are simply hedging a deflationary scenario that may not materialise in the end: however, we think that this methodology would allow the trader to macro-hedge his inflation book in a way that is consistent with his view of the economy and with

no model bias, given that the same model would be used to price the inflation client trade and the interest rates macro hedge.

19.3 Stress testing in the CTCB model

In recent years regulators have increasingly requested financial institutions to run stress tests: for example, the FED has introduced the CCAR in late 2010 (Comprehensive Capital Analysis and Review). One of the challenges that these financial institutions had to face was how to convert the market moves seen in the market and in the economy into the model parameter. Because the CTCB model takes the economy as an input, the economic shocks can be easily taken as an input and the model will answer to the questions asked by regulators: there is no need to shock model parameters like for example, given that the model calibration will deliver the new set of parameters that will fit to the stressed economic conditions.

20 Conclusions

In this article we have proposed a model inspired to the DSGE model to price inflation derivatives: in the recent years the debate around the central bank behaviour towards growth and inflation as been a pivotal topic.

Our modelling choice has offered new perspectives on inflation securities pricing and at the same time is extremely tractable, providing closed form solutions for the most common inflation and interest rates payoffs. Interestingly, we have showed that the short rate behaves according to the well-established Hull-White model. This makes the model calibration much simpler: we proposed a separable calibration strategy.

The good model tractability does not prevent one to use it for many pricing and risk applications, including stress tests and macr-hedging.

We are working to extend this framework in the following directions:

1. Calibrating the model to inflation and interest rates smiles;
2. Introducing credit risk and a multi-curve setting in this model;
3. Using this model to calculate counterparty and funding adjustments for OTC derivatives.

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